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IMMERSIONS OF HIGHLY CONNECTED MANIFOLDS

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ABSTRACT. If $[M^n \looparrowright \mathbf{R}^{2n-k}]$ denotes the set of regular homotopy classes of immersions, M^n a k-connected compact manifold, we show by a direct geometric construction the correspondence $[M^n \looparrowright \mathbf{R}^{2n-k}] \to [S^n \looparrowright \mathbf{R}^{2n-k}]$ for $0 \leq 2k \leq n-2$.

A bijection $I[M^n \looparrowright \mathbf{R}^{2n-k}] \to [S^n \looparrowright \mathbf{R}^{2n-k}]$ can be obtained as follows: Any immersion $g\colon M^n \looparrowright \mathbf{R}^{2n-k}$ is regularly homotopic to an immersion $\bar{g}$ with $\bar{g}|M \setminus U$ an embedding, where U is a suitable coordinate neighborhood of the basepoint. Set $I([g]) =$ the immersion of S^n determined by $\bar{g}|U$. That I is a bijection follows from [**2, 3**]. This paper gives a simple, direct geometric construction of the map $\bar{g}$.

PROPOSITION. *M^n a compact, closed, k-connected manifold. If $0 \leq 2k \leq n-2$, there is a bijection*

$$I\colon [M^n \looparrowright \mathbf{R}^{2n-k}] \to \Pi_n(V_n(\mathbf{R}^{2n-k})).$$

REMARK. The correspondence $[S^n \looparrowright \mathbf{R}^{2n-k}] \to \Pi_n(V_n(\mathbf{R}^{2n-k}))$ is a classical result of S. Smale [**7**]. Immersions of spheres are classified by their double points [**4, 6**]; this result extends this double point classification to highly connected manifolds.

For the proof, we first recall some well-known facts. By [**2**] a k-connected compact closed manifold M^n can be embedded in $\mathbf{R}^{2n-k}$ if $2k \leq n-2$. Moreover, two embeddings are regularly homotopic on $M \setminus \{x_0\}$. These results, together with [**7**], permit the construction of a group action

$$[S^n \looparrowright \mathbf{R}^{2n-k}] \times [M^n \looparrowright \mathbf{R}^{2n-k}] \to [M^n \looparrowright \mathbf{R}^{2n-k}]$$

which is seen to be free and transitive, hence the result. Let $g\colon M^n \looparrowright \mathbf{R}^{2n-k}$ represent a class $[g] \in [M^n \looparrowright \mathbf{R}^{2n-k}]$, and $f\colon S^n \looparrowright \mathbf{R}^{2n-k}$ represent an immersion with Smale invariant $c(f) \in \Pi_n(V_n(\mathbf{R}^{2n-k}))$. The connected sum of g and f yields an immersion $f \# g\colon M^n \looparrowright \mathbf{R}^{2n-k}$, and we define

$$[S^n \looparrowright \mathbf{R}^{2n-k}] \times [M^n \looparrowright \mathbf{R}^{2n-k}] \to [M^n \looparrowright \mathbf{R}^{2n-k}]$$

$$([f],[g]) \mapsto [f \# g].$$

It is obvious that in this way one obtains a group action. If g is as above, we may deform g by a regular homotopy such that

$$g \times g\colon M \times M \setminus \Delta_M \to \mathbf{R}^{2n-k} \times \mathbf{R}^{2n-k}$$

is transverse to the diagonal $\Delta \subset \mathbf{R}^{2n-k} \times \mathbf{R}^{2n-k}$.

The set $M_g = \{x \in M | \exists y \in M,\ x \neq y,\ g(x) = g(y)\}$ of double points of g is a k-dimensional submanifold if $2k \leq n-1$. The embedding $i\colon M_g \hookrightarrow M$ is null

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homotopic, so we may choose a homotopy $H\colon M_g \times I \to M$ from i to the trivial map $M_g \to x_0 \in M$. For a suitable coordinate neighbourhood $U \subset M^n$ of x_0 and a chart $k\colon (D^n, 0) \to (U, x_0)$ we can find $\epsilon > 0$ such that $H_\epsilon(M_g) \subset U$ for $t \in [1-\epsilon, 1]$. For $2k \leq n-2$ we can approximize H on $[0, 1-\epsilon/2]$ by $\overline{H}$, such that $\overline{H}_t\colon M_g \to M$ is an embedding. By a reparametrization we obtain an isotopy $\overline{H}\colon M_g \times I \to M$ with $\overline{H}_0 = i$ and $\overline{H}_1(M_g) \subset U$. Since M^n is closed, there is a diffeotopy $G\colon M \times I \to M$ such that

$$\begin{tikzcd} M_g \arrow[r, hook, "i"] \arrow[dr, "\overline{H}_t"'] & M \arrow[d, "G_t"] \\ & M \end{tikzcd}$$

commutes for all $t \in I$.

$g \circ G^{-1}$ is a regular homotopy from g to $\bar{g}$, and U contains the double points $M_{\bar{g}}$ of $\bar{g}$. $\bar{g}$ restricted to $M \setminus \mathring{U}$, ($\mathring{U}$ = interior of U) is an injective immersion of a compact manifold with boundary, hence an embedding. By [**5**] the regular homotopy class of $\bar{g}_{|M\setminus\mathring{U}}$ is unique.

$\bar{g} \circ K_{|\partial D^n}\colon S^{n-1} \to \mathbf{R}^{2n-k}$ is an embedding with Smale invariant zero, thus regularly homotopic to the standard embedding $S^{n-1} \hookrightarrow \mathbf{R}^{2n-k}$. Define $\tilde{g}\colon S^n \to \mathbf{R}^{2n-k}$ by $\bar{g} \circ k\colon D^n \to \mathbf{R}^{2n-k}$ on the northern hemisphere and a standard embedding on the southern hemisphere. Defining $I[g] := c[\tilde{g}]$, where c means take the Smale invariant, we obtain our map $I\colon [M^n \looparrowright \mathbf{R}^{2n-k}] \to [S^n \looparrowright \mathbf{R}^{2n-k}]$.

Claim. (i) I is well defined. (ii) I is bijective.

(i) Let f, g represent a class in $[M^n \looparrowright \mathbf{R}^{2n-k}]$. Proceeding as above we can regularly homotope f and g to an embedding outside a suitably chosen coordinate neighborhood of the basepoint of M. By [**2**] we may assume $f \mid M \setminus U = g \mid M \setminus U$ and a regular homotopy from f to g may be assumed to be an isotopy on $M \setminus U$. A regular homotopy from f to g is therefore given by a regular homotopy rel ∂U from $f \mid U$ to $g \mid U$. Applying the above gluing construction, one obtains a regular homotopy from f to g.

(ii) Let $j\colon M^n \hookrightarrow \mathbf{R}^{2n-k}$ be an embedding. An inverse J to I is given by $J(\varphi) = f_\varphi \# j$, where $f_\varphi\colon S^n \looparrowright \mathbf{R}^{2n-k}$ is an immersion with Smale invariant φ.

References

1. A. Haefliger, *Plongements differentiables des variétés dans variétés*, Comment. Math. Helv. **36** (1962).
2. A. Haefliger and M. W. Hirsch, *On the existence and classification of differentiable embeddings*, Topology **2** (1963).
3. M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.
4. U. Koschorke and B. Sanderson, *Geometric interpretation of the generalized Hopf invariant*, Math. Scand. **41** (1977).
5. M. Kervaire, *Sur l'invariant de Smale d'un plongement*, Comment. Math. Helv. **34** (1960).
6. C. Olk, *Immersionen von Mannigfaltigkeiten in euklidische Räume*, Dissertation, Siegen, 1980.
7. S. Smale, *The classification of immersions of spheres in Euclidean space*, Ann. of Math. (2) **69** (1959), 327–344.

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