TWO RESULTS CONCERNING CARDINAL FUNCTIONS ON COMPACT SPACES

I. JUHÁSZ AND Z. SZENTMIKLÓSSY

Abstract. We show that for $X$ compact $T_2$: (i) $d(X) \leq s(X) \cdot \hat{F}(X)$; (ii) if the pair $(\kappa, \hat{F}(X))$ is a caliber of $X$ then $\pi(X) < \kappa$.

These strengthen results of Šapirovskii from [3 and 5], respectively. Moreover, (i) settles a problem raised in [2] implying that there are no compact $T_2$ $\kappa$-examples for any singular cardinal $\kappa$.

In this note we follow the notation and terminology of [1]. In particular, we let $\hat{F}(X)$ denote the smallest cardinal $\kappa$ such that $|S| < \kappa$ for any free sequence $S \subseteq X$.

**Theorem 1.** If $X$ is compact $T_2$ then $d(X) \leq s(X) \cdot \hat{F}(X)$.

**Proof.** Let us put $s(X) \cdot \hat{F}(X) = \kappa$. Given any nonempty open set $U \subseteq X$ we can choose a family $\mathcal{C}(U)$ of open $F_\sigma$ sets in $X$ such that $U = \bigcup \mathcal{C}(U)$. But $X$ does not contain discrete subspaces of cardinality $\kappa^+$, hence, e.g. by 2.13 of [1], there is a subfamily $\mathcal{B}(U) \subseteq \mathcal{C}(U)$ and a subset $S(U) \subseteq U$ such that $|\mathcal{B}(U)| \leq \kappa$, $|S(U)| \leq \kappa$ and $U \subseteq \bigcup \mathcal{B}(U) \cup S(U)$.

Let us now assume, indirectly, that $d(X) > \kappa$. Then we also have $\pi(X) > \kappa$. Hence if $\mathcal{H}$ is a family of nonempty closed $G_\delta$ sets in $X$ with $|\mathcal{H}| \leq \kappa$, then there is an open nonempty $U \subseteq X$ such that $A \setminus U \neq \emptyset$ for each $A \in \mathcal{H}$. It follows easily from the compactness of $X$ that if $\mathcal{U}$ is a chain of open sets with this property, then $\bigcup \mathcal{U}$ possesses it as well. Thus by Zorn's lemma, we can fix an open set $W(H)$ which is maximal with respect to the above property. Observe that then for every nonempty set $H$ open in $X \setminus W(H)$, there is an $A \in \mathcal{H}$ with $A \subseteq H \cup W(H)$. Hence $\emptyset \neq A \setminus W(H) \subseteq H$, i.e. $\{A \setminus W(H) : A \in \mathcal{H}\}$ is a $\pi$-network in $X \setminus W(H)$. Consequently, we have

$$d(X \setminus W(H)) \leq |\mathcal{H}| \leq \kappa.$$ 

After these preparations we define by transfinite induction, families $\mathcal{B}_\alpha$ of closed $G_\delta$ subsets of $X$ for $\alpha \in \kappa$ with $|\mathcal{B}_\alpha| \leq \kappa$ as follows. If $\alpha \in \kappa$ and $\mathcal{B}_\beta$ has been defined for all $\beta < \alpha$, we consider the open set $W_\alpha = W\left(\bigcup \{\mathcal{B}_\beta : \beta < \alpha\}\right)$ and the family $\mathcal{B}(W_\alpha)$ of open $F_\sigma$ subsets of $W_\alpha$. For every $G \in \mathcal{B}(W_\alpha)$ we may then choose closed $G_\delta$ sets $F^\alpha_G$ for $n \in \omega$ such that $G = \bigcup \{F^\alpha_G : n \in \omega\}$. $\mathcal{B}_\alpha$ is then defined as the set of all nonempty finite intersections of members of the family

$$\bigcup \{\mathcal{B}_\beta : \beta < \alpha\} \cup \{X \setminus G : G \in \mathcal{B}(W_\alpha)\} \cup \{F^\alpha_G : G \in \mathcal{B}(W_\alpha), n \in \omega\}.$$

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Clearly $|S_\alpha| \leq \kappa$. This completes the induction.

Let us put

$$Y = \bigcup \{ S(W_\alpha) \cup (X \setminus W_\alpha) : \alpha \in \kappa \};$$

since $|S(W_\alpha)| \leq \kappa$ and $d(X \setminus W_\alpha) \leq \kappa$, then $d(Y) \leq \kappa$ as well; hence, by our indirect assumption, $X \neq Y$.

We may thus pick a point $p \in X \setminus Y$. Then for every $\alpha \in \kappa$ there are $G_\alpha \in \mathcal{B}(W_\alpha)$ and $n_\alpha \in \omega$ such that $p \in F_\alpha = F_{G_\alpha}^n \subset G_\alpha$. Let us put, for $\alpha \in \kappa$,

$$L_\alpha = \{ F_\beta : \beta \leq \alpha \} \cup \{ X \setminus G_\beta : \alpha < \beta < \kappa \}.$$

We claim that $L_\alpha$ is centered. Indeed, if $\mathcal{L} \in [L_\alpha]^\omega$, then $\bigcap \mathcal{L} \neq \emptyset$ follows by an easy induction on the number of the $X \setminus G_\beta$'s in $\mathcal{L}$ from our above construction.

Since the members of $L_\alpha$ are closed and nonempty, we can pick for each $\alpha \in \kappa$ a point $p_\alpha \in \bigcap L_\alpha$. But then $\{ p_\beta : \beta \in \alpha \} \subset X \setminus G_\alpha$ and $\{ p_\beta : \beta \in \kappa \setminus \alpha \} \subset F_\alpha$; hence

$$\{ p_\beta : \beta \in \alpha \} \cap \{ p_\beta : \beta \in \kappa \setminus \alpha \} = \emptyset.$$

This, however is a contradiction since $\{ p_\alpha : \alpha \in \kappa \}$ is a free sequence in $X$ of size $\kappa \geq \hat{\mathcal{F}}(X)$. Hence our proof is completed.

Theorem 1 is a strengthening of Šapirovskii's result saying that

$$d(X) \leq s(X) \cdot t(X),$$

for $X$ compact $T_2$, since, as is well known (see e.g. [1, 3.12]), for $X$ compact $T_2$ we have $F(X) = t(X)$. However the proofs of this given in [3, 4 or 1] do not seem to be modifiable to yield our result for the case in which $s(X) \cdot \hat{\mathcal{F}}(X) = \kappa$ is a singular cardinal. That this case is of some independent interest is shown by the following result that solves a problem raised in [2] (and answered there only partially even for the case $\text{cf}(\kappa) \leq \omega_1$).

**Corollary.** If $X$ is compact $T_2$, $\kappa$ is a singular cardinal, and $\pi(Y) < \kappa$ holds for each subspace $Y \subset X$ with $|Y| \leq \kappa$, then $\pi(X) < \kappa$ as well (or in the terminology of [2] there are no compact $T_2$ $\kappa$-examples).

**Proof.** Clearly $X$ may have no discrete subspaces of cardinality $\kappa$. Hence we have $d(X) \leq s(X) \cdot \hat{\mathcal{F}}(X) \leq \kappa$. But if $Y \subset X$ is dense with $|Y| \leq \kappa$, then by our assumption and 2.7 of [1], $\pi(X) = \pi(Y) < \kappa$.

To formulate our next result we recall that a pair $\langle \kappa, \lambda \rangle$ of cardinals is said to be a caliber of a space $X$ if for every family $\{ G_\xi : \xi \in \kappa \}$ of nonempty open sets in $X$ there is a set $A \subset \kappa$ with $|A| = \lambda$ such that $\bigcap \{ G_\xi : \xi \in A \} \neq \emptyset$.

**Theorem 2.** If $X$ is compact $T_2$ and the pair $\langle \kappa, \hat{\mathcal{F}}(X) \rangle$ is a caliber of $X$, then $\pi(X) < \kappa$.

**Proof.** Since the proof is quite similar to, but actually even simpler than, that of Theorem 1, we give only a sketch.

First, for any nonempty open set $U \subset X$ we fix a family $\mathcal{C}(U)$ of open $F_\alpha$'s in $X$ whose union is $U$. Second, assuming indirectly that $\pi(X) \geq \kappa$ and using that $\kappa > \omega$, for any family $\mathcal{A}$ of nonempty closed $G_\delta$'s with $|\mathcal{A}| < \kappa$, we pick a nonempty open $F_\alpha$.
set \( W(\alpha) \) such that \( A \setminus W(\alpha) \neq \emptyset \) for all \( A \in \mathcal{C} \). Then, by transfinite induction, families \( \mathfrak{B}_\alpha \) of nonempty closed \( G_\delta \) sets with \( |\mathfrak{B}_\alpha| \leq |\alpha| + \omega < \kappa \) are defined for \( \alpha \in \kappa \) as follows. If \( \alpha \in \kappa \) and \( \mathfrak{B}_\beta \) have been chosen for \( \beta \in \alpha \), put

\[
W_\alpha = W\left( \bigcup \{\mathfrak{B}_\beta : \beta \in \alpha\} \right).
\]

We can write

\[
W_\alpha = \bigcup \{F_\alpha^n : n \in \omega\},
\]

where the \( F_\alpha^n \) are closed \( G_\delta \) sets in \( X \). Now, \( \mathfrak{B}_\alpha \) is defined as the set of all nonempty finite intersections of members of the family

\[
\bigcup \{\mathfrak{B}_\beta : \beta \in \alpha\} \cup \{X \setminus W_\alpha\} \cup \{F_\alpha^n : n \in \omega\}.
\]

Clearly, \( |\mathfrak{B}_\alpha| \leq |\alpha| + \omega \).

Considering the family \( \{W_\alpha : \alpha \in \kappa\} \) and using that, with \( \lambda = \check{\pi}(X) \), the pair \( \langle \kappa, \lambda \rangle \) is a caliber of \( X \), we can find a set \( A \subset \kappa \) with \( |A| = \lambda \) such that

\[
\bigcap \{W_\alpha : \alpha \in A\} \neq \emptyset.
\]

Let \( p \in \bigcap \{W_\alpha : \alpha \in A\} \), and for each \( \alpha \in A \) choose \( n_\alpha \in \omega \) such that \( p \in F_\alpha^{n_\alpha} = F_\alpha \subset W_\alpha \). Exactly as in the proof of Theorem 1 we can see that for \( \alpha \in A \) the family

\[
\mathcal{L}_\alpha = \{F_\beta : \beta \in A \& \beta \leq \alpha\} \cup \{X \setminus W_\beta : \beta \in A \& \alpha < \beta\}
\]

is centered, and if \( p_\alpha \in \bigcap \mathcal{L}_\alpha \) for \( \alpha \in A \), then \( \{p_\alpha : \alpha \in A\} \) is a free sequence in \( X \) of size \( \lambda = \check{\pi}(X) \), a contradiction. This completes the proof.

In [5] Šapirovskii proved that if \( t(X)^+ \) is a caliber of a compact \( T_2 \) space \( X \) then \( \pi(X) \leq t(X) \). Since \( \check{\pi}(X) \leq F(X)^+ = t(X)^+ \) and, moreover, if \( F(X)^+ \) is a caliber of \( X \), clearly so is the pair \( \langle F(X)^+, \check{F}(X) \rangle \) as well, this result is an immediate corollary of Theorem 2.

REFERENCES


MATHEMATICAL INSTITUTE, HUNGARIAN ACADEMY OF SCIENCES, RÉÁLTANODA 13–15, BUDAPEST 1479 PF 128, HUNGARY