A TOPOLOGICAL SPACE WITHOUT A COMPLETE QUASI-UNIFORMITY

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Abstract. We show that an example of Burke and van Douwen has no complete quasi-uniformity. Moreover, we show that it is almost finitely-fully normal but not almost $S_0$-fully normal.

0. Introduction. Every topological space admits a quasi-uniformity. The problem whether every topological space admits a complete quasi-uniformity is considered in [3, Problem C], where an example is given of a $T_1$-space that admits a complete, but no convergence complete, quasi-uniformity. In this note we show that a locally compact separable normal $M$-space of D. K. Burke and E. K. van Douwen admits no complete quasi-uniformity, thereby answering an old question in the theory of quasi-uniform spaces. Moreover, we show that this space is an almost finitely-fully normal countably paracompact space that is not almost $S_0$-fully normal. It is interesting to compare these results with the recent results of K. P. Hart [4] that M. E. Rudin’s Dowker space is both orthocompact and finitely-fully normal; for it follows readily from Hart’s results that, while Rudin’s space is not almost $K_0$-fully normal, it does admit a complete quasi-uniformity.

1. Definitions and a lemma. A quasi-uniformity on a set $X$ is a filter $\mathfrak{U}$ on $X \times X$ such that (a) each member of $\mathfrak{U}$ is a reflexive relation on $X$, and (b) if $U \in \mathfrak{U}$ then $V \circ V \subseteq U$ for some $V \in \mathfrak{U}$. The pair $(X, \mathfrak{U})$ is called a quasi-uniform space. A filter $\mathfrak{F}$ on $(X, \mathfrak{U})$ is a Cauchy filter provided that for each $U \in \mathfrak{U}$ there exists $p \in X$ so that $U(p) \in \mathfrak{F}$, and $(X, \mathfrak{U})$ is said to be complete provided that every Cauchy filter has a cluster point. The topology $\tau(\mathfrak{U}) = \{G \subseteq X: \text{for each } x \in G \text{ there is } U \in \mathfrak{U} \text{ with } U(x) \subseteq G\}$ is called the topology induced by $\mathfrak{U}$. A topological space $(X, \tau)$ admits $\mathfrak{U}$ provided that $\tau$ is the topology induced by $\mathfrak{U}$. Let $(X, \tau)$ be a topological space and let $\mathfrak{F}$ be the collection of reflexive transitive relations $V$ on $X$ for which $V(x) \subseteq \tau$ for all $x \in X$. Then $\mathfrak{F}$ is a filterbase for a quasi-uniformity $\mathfrak{U}$. Moreover, using the observation of W. J. Pervin [8] that for each open set $G, G \times G \cup (X \setminus G) \times X \in \mathfrak{U}$, we see that $(X, \tau)$ admits $\mathfrak{U}$. It is known that this quasi-uniformity $\mathfrak{U}$ is complete if and only if every ultrafilter on $X$ without a cluster point has a closure-preserving subcollection without a cluster point [3, p. 59]. Consequently, a

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regular space that is either almost real-compact or weakly orthocompact admits a complete quasi-uniformity.

Throughout, if \( \mathcal{R} \) is a cover of a space \( X \), and \( A \) is a subset of some member of \( \mathcal{R} \), we say that \( A \) is a refiner of \( \mathcal{R} \). A space \( X \) is almost finitely-fully normal (almost \( S_0 \)-fully normal) \([6]\) provided that if \( \mathcal{C} \) is an open cover of \( X \) there is an open refinement \( \mathcal{R} \) of \( \mathcal{C} \) with the property that if \( M \) is a finite (countable) set and \( M \) is a refiner of \( \{st(x, \mathcal{R}) \mid x \in X\} \), then \( M \) is also a refiner of \( \mathcal{C} \).

We begin with a slight extension of a result of G. Aquaro [1] and K. Morita [7, Lemma 4.3].

**Lemma 1.** A space \( X \) is almost finitely-fully normal (almost \( S_0 \)-fully normal) if and only if \( X \) is normal, and for each open cover \( \mathcal{C} \) of \( X \) there is a locally finite open cover \( \mathcal{R} \) of \( X \) so that if \( M \) is a refiner of \( \mathcal{R} \) and \( M \) is a finite (countable) set, then \( M \) is a refiner of \( \mathcal{C} \).

**2. The example.** The example \( X \) under consideration is described completely in [2]. For our purposes it is enough to know the following details: The ground set is \( \mu \cup (\omega \times \omega) \), \( \mu \) and \( \omega \) considered disjoint, where \( \mu \) is a regular cardinal, and there is a collection \( F = \{f_\alpha : \alpha \in \mu\} \) such that

1. For each \( \alpha \in \mu \), \( f_\alpha \in \omega^\omega \).
2. Each \( f_\alpha \) is nondecreasing.
3. There is no \( g \in \omega^\omega \) so that \( f_\alpha < *g \) for all \( \alpha \in \mu \).

(As usual, we say \( f < *g \) provided that for all but finitely many \( n \in \omega \), \( f(n) < g(n) \).)

The points of \( \omega \times \omega \) are isolated, and basic open sets about \( \alpha \in \mu \) with \( 0 \leq \beta < \alpha < \mu \) and \( m \in \omega \) are of the form
\[
U(\alpha, \beta, m) = (\beta, \alpha] \cup \{\langle k, n \rangle : k \geq m \text{ and } f_\beta(k) < n \leq f_\alpha(k)\}.
\]
If \( \alpha \in \mu \) and \( \alpha = 0 \), basic open sets about \( \alpha \) are of the form
\[
U(0, \beta, m) = \{0\} \cup \{\langle k, n \rangle : k \geq m, n \leq f_0(k)\} \quad \text{where } m \in \omega.
\]

**Lemma 2.** If \( S \) is a cofinal subset of \( \mu \), then \( A = \{k \in \omega : \langle f(k)\rangle_{s \in S} \text{ is eventually bounded}\} \) is an initial segment of \( \omega \).

**Proof.** Since each \( f_\alpha \) is nondecreasing, we note that if \( a < b < \omega \) and \( b \in A \), then \( a \in A \). Assume \( A = \omega \). Then for all \( n \in \omega \) there are \( s_n \in S \) and \( k_n \in \omega \) such that if \( s \in S \) and \( s > s_n \) then \( f_s(n) < k_n \). Define \( g : \omega \rightarrow \omega \) by \( g(n) = k_n \). There is a tail \( S' \) of \( S \) such that if \( s \in S' \) then \( f_s(n) < g(n) \) for all \( n \in \omega \). Let \( f_a \in F \). There is an \( s \in S' \) with \( \alpha < s \); by (2), \( f_a < *f_s \), and thus \( f_a < g \). We have shown that (3) fails—a contradiction. \( \blacksquare \)

\( X \) admits no complete quasi-uniformity. For each \( x \in \mu \) and \( m \in \omega \), set \( F(x, m) = \{\langle k, n \rangle : k \geq m \text{ and } f_x(k) < n\} \) and let \( \mathcal{F} \) be the filter for which \( \{F(x, m) : x \in \mu \text{ and } m \in \omega\} \) is a filter base. Clearly no point of \( \omega \times \omega \) is a cluster point of \( \mathcal{F} \). If \( \rho \in \mu \) and \( m \in \omega \), then \( U(\rho, 0, 0) \cap F(\rho, m) = \emptyset \). Therefore \( \mathcal{F} \) is a filter without a cluster point. We show that \( \mathcal{F} \) is a Cauchy filter with respect to each quasi-uniformity that \( X \) admits. Let \( V \) be such a quasi-uniformity, let \( V \in \mathcal{F} \) and let \( W \in \mathcal{F} \) so \( W^2 \subset V \). For
each $x \in \mu$ choose $\beta_x \in \mu$ and $m_x \in \omega$ so that $U(x, \beta_x, m_x) \subset W(x)$. By the Pressing-Down Lemma, there is a cofinal subset $S$ of $\mu$, $\beta \in \mu$ and $j \in \omega$ so that, for all $s \in S$, $U(s, \beta, j) \subset W(s)$. We note that $\beta < s$ for each $s \in S$. Set $A' = \{ k \in \omega: \langle f_s(k) \rangle_{s \in S} \text{ is bounded} \}$. By Lemma 2, $A'$ is finite. There is $e \in \omega$ so that if $k \geq e$ then $\{ f_s(k) : s \in S \}$ is unbounded. Let $r = \max\{ e, j \}$. For each $k \geq r$, define a function $a_k: \omega \to S$ by letting $a_k(n)$ be the least ordinal $\alpha \in S$ so $f_\alpha(k) \geq n$. Let $\gamma = \sup\{ a_k(n) : k \geq e, n \in \omega \}$; there exists $s_0 \in S$ with $\gamma < s_0 < \mu$. Let $k \geq r$ and let $m \in \omega$. Then $a_k(m) \in U(s_0, \beta, j) \subset W(s_0)$ so $W(a_k(m)) \subset W^2(s_0) \subset V(s_0)$. We show that $F(\beta, r) \subset V(s_0)$ so $V(s_0) \subset \mathcal{W}$ as required. Let $\langle k, n \rangle \in F(\beta, r)$. Then $k \geq r \geq j$ and $f_\beta(k) < n \subset \mathcal{W}$. Then $\langle k, n \rangle \in U(a_k(n), \beta, j) \subset W(a_k(n)) \subset V(s_0)$. 

**Corollary.** The space $X$ is not weakly orthocompact. (Indeed, $X$ is a transitive space that is not preorthocompact; see [3, Lemma 6.16 and Corollary 5.11].)

For the nonce, mimicking terminology of E. Hewitt, we say a topological space is $q$-complete provided it admits a complete quasi-uniformity. In this terminology, Proposition 3.12 of [3] shows that the perfect preimage of a $q$-complete space is $q$-complete. The present example shows that this result does not obtain if perfect maps are replaced by quasi-perfect maps.

**3. Further properties of the example.** $X$ is almost finitely-fully normal. Let $\mathcal{B}$ be an open cover of $X$. Without loss of generality, we assume that for each $x \in \mu$ there is $B_x \in \mathcal{B}$ of the form $B_x = U(x, \beta_x, m_x)$ and that all remaining members of $\mathcal{B}$ are isolated points. Since $\mu$ is a regular cardinal, by the Pressing-Down Lemma, there are a cofinal subset $S$ of $\mu$, $\beta \in \mu$, and $m \in \omega$ so that $\beta_x = \beta$ and $m_x = m$ for all $x \in S$.

Set $S = \{ k \in \omega: \langle f_s(k) \rangle_{s \in S} \text{ is eventually bounded} \}$. By Lemma 2, $S$ is a finite set. Let $h$ be a natural number exceeding $\max(A)$. Set $R = (\beta, \mu) \cup \{ \langle k, n \rangle: k \geq h, m \text{ and } f_\beta(k) < n \}$. Then $R$ is an open set, and since $\mu \setminus R$ is compact there is a finite subset $R'$ of $\mathcal{B}$ so that $\mu \setminus R \subset \cup R'$. Let

$$
\mathcal{R} = \mathcal{B}' \cup \{ R \} \cup \{ \{ x \}: x \not\in \cup (\mathcal{B}' \cup \{ R \}) \}.
$$

Then $\mathcal{R}$ is a locally finite open cover of $X$. Let $M$ be a finite refiner of $\mathcal{R}$. By Lemma 1 it suffices to show that $M$ is a refiner of $\mathcal{B}$. We assume $M \subset R$. Set $M_1 = M \cap \mu$ and set $M_2 = M \cap (\omega \times \omega)$. List the members of $M_2$ as $\langle k_1, n_1 \rangle$, $\langle k_2, n_2 \rangle$, ..., $\langle k_j, n_j \rangle$, where $k_1 < k_2 < \cdots < k_j$. Let $q = \max\{ n_1, n_2, \ldots, n_j \}$. There is an $s' \in S$ so that $s' > \max M_1$. Since $k_1 \geq h$, $k_1 \not\in A$ so there is an $s \in S$ with $s > s'$ so $f_s(k_1) > q$. Let $1 \leq d \leq j$. Since $M \subset R$, $f_\beta(k_d) < n_d \leq q < f_s(k_1) \leq f_s(k_d)$. Thus $M_2 \subset \{ \langle k, n \rangle: k \geq m, f_\beta(k) < n \leq f_s(k) \} \subset U(s, \beta, m)$. It follows that $M \subset U(s, \beta, m)$.

$X$ is not almost $\aleph_0$-fully normal. Set $\mathcal{B}' = \{ U(x, 0, 0): x \in \mu \}$ and set $\mathcal{B} = \mathcal{B}' \cup \{ \{ x \}: x \in \omega \times \omega \}$. Suppose $\mathcal{L}$ is an open locally finite cover of $X$ with the property that every countable refiner of $\mathcal{L}$ is a refiner of $\mathcal{B}$. Since $\mathcal{L} \setminus \mu$ is locally finite, and hence finite, only finitely many members of $\mathcal{L}$ meet $\mu$; list these members as $L_1, L_2, \ldots, L_r$. For each $x \in \mu$, there are $k_x, \beta_x$ and $n_x$ so that $x \in U(x, \beta_x, n_x) \subset L_{k_x}$.
where \( U(x, \beta_x, n_x) \) is a basic open set about \( x \). By the Pressing-Down Lemma, there is a cofinal subset \( S \) of \( \mu \), \( m \in \omega \), \( \tilde{k} \) with \( 1 \leq \tilde{k} \leq z \), and \( \tilde{\beta} \in \mu \) so that for each \( x \in S \), \( k_x = \tilde{k} \), \( \beta_x = \tilde{\beta} \) and \( n_x = m \). Set \( \tilde{A} = \{ k \in \omega : \langle f(k) \rangle_{x \in S} \text{ is eventually bounded} \} \). Then \( \tilde{A} \) is finite and there is a natural number \( \tilde{h} \) exceeding \( \max(\tilde{A}) \). Set 
\[
\tilde{R} = (\tilde{\beta}, \mu) \cup \{ \langle k, n \rangle : k \geq \tilde{h}, m \text{ and } f_{\tilde{\beta}}(k) < n \}.
\]
Then \( \tilde{R} \subset L_z^\omega \) so each countable subset of \( \tilde{R} \) is a refiner of \( \tilde{R} \). Let \( D = \tilde{R} \setminus \mu \). Then \( D \)
is a refiner of \( \tilde{R} \)—a contradiction. ■

In his thesis [5, Theorem 2.2.10], H. J. K. Junnila proves that a space \( X \) is 2-fully normal if and only if it is almost 2-fully normal and for each open cover \( C \) of \( X \) there is a reflexive relation \( V \) on \( X \) such that, for each \( x \in X \), \( V(x) \) is open and such that, for each \( x \in X \) and \( y \in V(x) \), \( V(x) \cup V(y) \) is a refiner of \( C \). In particular, every orthocompact almost 2-fully normal space is 2-fully normal. It is unknown whether the converse holds; the referee suggests that the space we have considered above could possibly be used in the construction of a counterexample.

Since the space \( X \) was constructed as an example of a space that does not have a countably-compactification, it is interesting to note that nearly the same method of proof establishes that the following countably compact normal space \( Y \) is not \( q \)-complete.

Let \( \langle A_\alpha \rangle_{\alpha \in \mu} \) be an increasing maximal tower on \( \omega \), where \( \mu \) is a regular cardinal. Let \( Y = \mu \cup \omega \) and, as usual, define a topology on \( Y \) by specifying the following neighborhoods: Points of \( \omega \) are isolated. If \( 0 \leq \beta < \alpha < \mu \) and \( F \) is a finite subset of \( \omega \), set
\[
U(\alpha, \beta, F) = (\beta, \alpha] \cup [A_\alpha \setminus A_\beta] \setminus F,
\]
and if \( \alpha = 0 \) and \( F \) is a finite subset of \( \omega \), set
\[
U(0, \beta, F) = \{0\} \cup (A_0 \setminus F).
\]
Then \( \mathcal{G} = \text{fil}\{ (\omega \setminus A_\alpha) \setminus F : \alpha \in \mu \text{ and } F \text{ is a finite subset of } \omega \} \) is a filter without a cluster point that is a Cauchy filter with respect to each quasi-uniformity that \( Y \) admits.

The similarity of the methods of proof that \( X \) and \( Y \) are not \( q \)-complete is not just a coincidence. The basic neighborhoods of points of \( \mu \) in \( X \) can be defined in terms of the following tower on \( \omega \times \omega : \langle \{ (k, n) : k \in \omega, n = f_{\alpha}(k) \} \rangle_{\alpha \in \mu} \).

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