A TOPOLOGICAL SPACE WITHOUT A COMPLETE QUASI-UNIFORMITY

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Abstract. We show that an example of Burke and van Douwen has no complete quasi-uniformity. Moreover, we show that it is almost finitely-fully normal but not almost $\mathcal{N}_0$-fully normal.

0. Introduction. Every topological space admits a quasi-uniformity. The problem whether every topological space admits a complete quasi-uniformity is considered in [3, Problem C], where an example is given of a $T_1$-space that admits a complete, but no convergence complete, quasi-uniformity. In this note we show that a locally compact separable normal $M$-space of D. K. Burke and E. K. van Douwen admits no complete quasi-uniformity, thereby answering an old question in the theory of quasi-uniform spaces. Moreover, we show that this space is an almost finitely-fully normal countably paracompact space that is not almost $\mathcal{N}_0$-fully normal. It is interesting to compare these results with the recent results of K. P. Hart [4] that M. E. Rudin's Dowker space is both orthocompact and finitely-fully normal; for it follows readily from Hart's results that, while Rudin's space is not almost $\mathcal{K}_0$-fully normal, it does admit a complete quasi-uniformity.

1. Definitions and a lemma. A quasi-uniformity on a set $X$ is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ is a reflexive relation on $X$, and (b) if $U \in \mathcal{U}$ then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is called a quasi-uniform space. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is a Cauchy filter provided that for each $U \in \mathcal{U}$ there exists $p \in X$ so that $U(p) \in \mathcal{F}$, and $(X, \mathcal{U})$ is said to be complete provided that every Cauchy filter has a cluster point. The topology $\tau(\mathcal{U}) = \{G : \text{for each } x \in G \text{ there is } U \in \mathcal{U} \text{ with } U(x) \subseteq G\}$ is called the topology induced by $\mathcal{U}$. A topological space $(X, \tau)$ admits $\mathcal{U}$ provided that $\tau$ is the topology induced by $\mathcal{U}$. Let $(X, \tau)$ be a topological space and let $\mathcal{B}$ be the collection of reflexive transitive relations $V$ on $X$ for which $V(x) \subseteq \tau$ for all $x \in X$. Then $\mathcal{B}$ is a filterbase for a quasi-uniformity $\mathcal{U}$. Moreover, using the observation of W. J. Pervin [8] that for each open set $G$, $G \times G \cup (X \setminus G) \times X \in \mathcal{U}$, we see that $(X, \tau)$ admits $\mathcal{U}$. It is known that this quasi-uniformity $\mathcal{U}$ is complete if and only if every ultrafilter on $X$ without a cluster point has a closure-preserving subcollection without a cluster point [3, p. 59]. Consequently, a

1Received by the editors February 14, 1983 and, in revised form, June 28, 1983.
1980 Mathematics Subject Classification. Primary 54E15, 54D20.
Key words and phrases. Almost $\mathcal{N}_0$-fully normal, almost finitely-fully normal, complete quasi-uniformity.

1While working on this paper, the first author was supported by the Schweizerischer Nationalfonds.

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regular space that is either almost real-compact or weakly orthocompact admits a complete quasi-uniformity.

Throughout, if $\mathcal{R}$ is a cover of a space $X$, and $A$ is a subset of some member of $\mathcal{R}$, we say that $A$ is a refiner of $\mathcal{R}$. A space $X$ is almost finitely-fully normal (almost $\mathfrak{S}_{\omega}$-fully normal) [6] provided that if $\mathcal{C}$ is an open cover of $X$ there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ with the property that if $M$ is a finite (countable) set and $M$ is a refiner of $\{st(x,\mathcal{R})|x \in X\}$, then $M$ is also a refiner of $\mathcal{C}$.

We begin with a slight extension of a result of G. Aquaro [1] and K. Morita [7, Lemma 4.3].

**Lemma 1.** A space $X$ is almost finitely-fully normal (almost $\mathfrak{S}_{\omega}$-fully normal) if and only if $X$ is normal, and for each open cover $\mathcal{C}$ of $X$ there is a locally finite open cover $\mathcal{R}$ of $X$ so that if $M$ is a refiner of $\mathcal{R}$ and $M$ is a finite (countable) set, then $M$ is a refiner of $\mathcal{C}$.

**2. The example.** The example $X$ under consideration is described completely in [2]. For our purposes it is enough to know the following details: The ground set is $\mu \cup (\omega \times \omega)$, $\mu$ and $\omega$ considered disjoint, where $\mu$ is a regular cardinal, and there is a collection $F = \{f_\alpha: \alpha \in \mu\}$ such that

1. for each $\alpha \in \mu$, $f_\alpha \in \omega \omega$,
2. each $f_\alpha$ is nondecreasing,
3. $f_\alpha < *f_\beta$ if $\alpha < \beta$, 
4. there is no $g \in \omega \omega$ so that $f_\alpha \leq *g$ for all $\alpha \in \mu$.

(As usual, we say $f < *g$ provided that for all but finitely many $n \in \omega$, $f(n) < g(n)$.)

The points of $\omega \times \omega$ are isolated, and basic open sets about $\alpha \in \mu$ with $0 \leq \beta < \alpha < \mu$ and $m \in \omega$ are of the form

$$U(\alpha, \beta, m) = (\beta, \alpha] \cup \{\langle k, n \rangle: k \geq m \text{ and } f_\alpha(k) < n \leq f_\beta(k)\}.$$

If $\alpha \in \mu$ and $\alpha = 0$, basic open sets about $\alpha$ are of the form

$$U(0, \beta, m) = \{0\} \cup \{\langle k, n \rangle: k \geq m, n \leq f_0(k)\} \text{ where } m \in \omega.$$

**Lemma 2.** If $S$ is a cofinal subset of $\mu$, then $A = \{k \in \omega: \langle f(k)\rangle_{s \in S} \text{ is eventually bounded}\}$ is an initial segment of $\omega$.

**Proof.** Since each $f_\alpha$ is nondecreasing, we note that if $a < b < \omega$ and $b \in A$, then $a \in A$. Assume $A = \omega$. Then for all $n \in \omega$ there are $s_n \in S$ and $k_n \in \omega$ such that if $s \in S$ and $s > s_n$ then $f_s(n) < k_n$. Define $g: \omega \to \omega$ by $g(n) = k_n$. There is a tail $S'$ of $S$ such that if $s \in S'$ then $f_s(n) < g(n)$ for all $n \in \omega$. Let $f_\alpha \in F$. There is an $s \in S'$ with $\alpha < s$; by (2), $f_\alpha \leq *f_s$, and thus $f_\alpha \leq g$. We have shown that (3) fails—a contradiction.

$X$ admits no complete quasi-uniformity. For each $x \in \mu$ and $m \in \omega$, set $F(x, m) = \{\langle k, n \rangle: k \geq m \text{ and } f_\alpha(k) < n\}$ and let $\mathcal{F}$ be the filter for which $\{F(x, m): x \in \mu \text{ and } m \in \omega\}$ is a filter base. Clearly no point of $\omega \times \omega$ is a cluster point of $\mathcal{F}$. If $p \in \mu$ and $m \in \omega$, then $U(p, 0, 0) \cap F(p, m) = \emptyset$. Therefore $\mathcal{F}$ is a filter without a cluster point. We show that $\mathcal{F}$ is a Cauchy filter with respect to each quasi-uniformity that $X$ admits. Let $V \in \mathcal{V}$ be such a quasi-uniformity, let $V \in \mathcal{V}$ and let $W \in \mathcal{V}$ so $W^2 \subset V$. For
each $x \in \mu$ choose $\beta_x \in \mu$ and $m_x \in \omega$ so that $U(x, \beta_x, m_x) \subseteq W(x)$. By the
Pressing-Down Lemma, there is a cofinal subset $S$ of $\mu$, $\beta \in \mu$ and $j \in \omega$ so that, for
all $s \in S$, $U(s, \beta, j) \subseteq W(s)$. We note that $\beta < s$ for each $s \in S$. Set $A' = \{k \in \omega:
\langle f_s(k) \rangle_{s \in S}$ is bounded}. By Lemma 2, $A'$ is finite. There is $e \in \omega$ so that if $k \geq e$
then $\{|f_s(k)| : s \in S\}$ is unbounded. Let $r = \max\{e, j\}$. For each $k \geq r$, define a
function $a_k : \omega \to S$ by letting $a_k(n)$ be the least ordinal $\alpha \in S$ so $f_s(k) \geq \alpha$. Let
$\gamma = \sup\{a_k(n) : k \geq r, n \in \omega\}$; there exists $s_0 \in S$ with $\gamma < s_0 < \mu$. Let $k \geq r$ and let
$m \in \omega$. Then $a_k(m) \in U(s_0, \beta, j) \subseteq W(s_0)$. We show that $F(\beta, j) \subseteq V(s_0)$
so $V(s_0) \subseteq f$ as required. Let $\langle k, n \rangle \in F(\beta, r)$. Then
\[ \langle k, n \rangle \in U(a_k(n), \beta, j) \subseteq W(a_k(n)) \subseteq V(s_0) \]  

**Corollary.** The space $X$ is not weakly orthocompact. (Indeed, $X$ is a transitive
space that is not preorthocompact; see [3, Lemma 6.16 and Corollary 5.11].)

For the nonce, mimicking terminology of E. Hewitt, we say a topological space is
$q$-complete provided it admits a complete quasi-uniformity. In this terminology,
Proposition 3.12 of [3] shows that the perfect preimage of a $q$-complete space is
$q$-complete. The present example shows that this result does not obtain if perfect
maps are replaced by quasi-perfect maps.

**3. Further properties of the example.** $X$ is almost finitely-fully normal. Let $\mathcal{B}$ be an
open cover of $X$. Without loss of generality, we assume that for each $x \in \mu$ there is
$B_x \in \mathcal{B}$ of the form $B_x = U(x, \beta_x, m_x)$ and that all remaining members of $\mathcal{B}$ are
isolated points. Since $\mu$ is a regular cardinal, by the Pressing-Down Lemma, there are
a cofinal subset $S$ of $\mu$, $\beta \in \mu$, and $m \in \omega$ so that $\beta_x = \beta$ and $m_x = m$ for all $x \in S$.

Set $A = \{k \in \omega : \langle f_s(k) \rangle_{s \in S}$ is eventually bounded}. By Lemma 2, $A$ is a finite
set. Let $h$ be a natural number exceeding $\max(A)$. Set $R = (\beta, \mu) \cup \langle \langle k, n \rangle : k \geq h, m$
and $f_\beta(k) < n \rangle$. Then $R$ is an open set, and since $\mu \setminus R$ is compact there is a
finite subset $\mathcal{R}'$ of $\mathcal{B}$ so that $\mu \setminus R \subseteq \mathcal{B}'$. Let
\[ \mathcal{R} = \mathcal{R}' \cup \{R\} \cup \{\{x\} : x \in \omega \times \omega \} \] 

Then $\mathcal{R}$ is a locally finite open cover of $X$. Let $M$ be a finite refiner of $\mathcal{R}$. By Lemma
1 it suffices to show that $M$ is a refiner of $\mathcal{B}$. We assume $M \subseteq R$. Set $M_1 = M \cap \mu
and set $M_2 = M \cap (\omega \times \omega)$. List the members of $M_2$ as $\langle k_1, n_1 \rangle, \langle k_2, n_2 \rangle, \ldots,
\langle k_j, n_j \rangle$, where $k_1 < k_2 \leq \cdots \leq k_j$. Let $q = \max\{n_1, n_2, \ldots, n_j\}$. There is an $s' \in S
so that $s' > \max M_1$. Since $k_1 \geq h$, $k_1 \notin A$ so there is an $s \in S$ with $s > s'$ so
$f_\beta(k_1) > q$. Let $1 \leq d \leq j$. Since $M \subseteq R$, $f_\beta(k_d) < n_d \leq q < f_\beta(k_1) < f_\beta(k_d)$. Thus
$M_2 \subseteq \{\langle k, n \rangle : k \geq m, f_\beta(k) < n \leq f_\beta(k) \subseteq U(s, \beta, m)$. It follows that $M \subseteq U(s, \beta, m)$. $X$ is not almost $\mathcal{N}_0$fully normal. Set $\mathcal{B}' = \{U(x, 0, 0) : x \in \mu\}$ and set $\mathcal{B} = \mathcal{B}' \cup
\{\{x\} : x \in \omega \times \omega\}$. Suppose $\mathcal{L}$ is an open locally finite cover of $X$ with the property
that every countable refiner of $\mathcal{L}$ is a refiner of $\mathcal{B}$. Since $\mathcal{L} \setminus \mu$ is locally finite, and
hence finite, only finitely many members of $\mathcal{L}$ meet $\mu$; list these members as $L_1,
L_2, \ldots, L_j$. For each $x \in \mu$, there are $k_x, \beta_x$ and $n_x$ so that $x \in U(x, \beta_x, n_x) \subseteq L_{k_x}$,
where $U(x, \beta_x, n_x)$ is a basic open set about $x$. By the Pressing-Down Lemma, there is a cofinal subset $S$ of $\mu$, $\tilde{m} \in \omega$, $\tilde{k}$ with $1 \leq \tilde{k} \leq z$, and $\tilde{\beta} \in \mu$ so that for each $x \in S$, $k_x = \tilde{k}$, $\beta_x = \tilde{\beta}$ and $n_x = \tilde{m}$. Set $\tilde{A} = \{k \in \omega: \langle f_{j}(k) \rangle_{j \in \tilde{S}}$ is eventually bounded$\}$. Then $\tilde{A}$ is finite and there is a natural number $\tilde{h}$ exceeding $\max(\tilde{A})$. Set

$$\tilde{R} = (\tilde{\beta}, \tilde{m}) \cup \{\langle k, n \rangle: k \geq \tilde{h}, \tilde{m} \text{ and } f_{\tilde{\beta}}(k) < n\}.$$ 

Then $\tilde{R} \subseteq \mathcal{L}^{-1}$ so each countable subset of $\tilde{R}$ is a refiner of $\mathcal{R}$. Let $D = \tilde{R} \setminus \mu$. Then $D$ is a refiner of $\mathcal{R}$—a contradiction. 

In his thesis [5, Theorem 2.2.10], H. J. K. Junnila proves that a space $X$ is 2-fully normal if and only if it is almost 2-fully normal and for each open cover $\mathcal{C}$ of $X$ there is a reflexive relation $V$ on $X$ such that, for each $x \in X$, $V(x)$ is open and such that, for each $x \in X$ and $y \in V(x)$, $V(x) \cup V(y)$ is a refiner of $\mathcal{C}$. In particular, every orthocompact almost 2-fully normal space is 2-fully normal. It is unknown whether the converse holds; the referee suggests that the space we have considered above could possibly be used in the construction of a counterexample.

Since the space $X$ was constructed as an example of a space that does not have a countably-compactification, it is interesting to note that nearly the same method of proof establishes that the following countably compact normal space $Y$ is not $q$-complete.

Let $\langle A_{\alpha} \rangle_{\alpha \in \mu}$ be an increasing maximal tower on $\omega$, where $\mu$ is a regular cardinal. Let $Y = \mu \cup \omega$ and, as usual, define a topology on $Y$ by specifying the following neighborhoods: Points of $\omega$ are isolated. If $0 < \beta < \alpha < \mu$ and $F$ is a finite subset of $\omega$, set

$$U(\alpha, \beta, F) = (\beta, \alpha] \cup [A_{\alpha} \setminus A_{\beta}] \setminus F,$$

and if $\alpha = 0$ and $F$ is a finite subset of $\omega$, set

$$U(0, \beta, F) = \{0\} \cup (A_{0} \setminus F).$$

Then $\mathcal{G} = \text{fil}\{(\omega \setminus A_{\alpha}) \setminus F: \alpha \in \mu \text{ and } F \text{ is a finite subset of } \omega\}$ is a filter without a cluster point that is a Cauchy filter with respect to each quasi-uniformity that $Y$ admits.

The similarity of the methods of proof that $X$ and $Y$ are not $q$-complete is not just a coincidence. The basic neighborhoods of points of $\mu$ in $X$ can be defined in terms of the following tower on $\omega \times \omega$: $\langle\langle k, n \rangle: k \in \omega, n \leq f_{\alpha}(k)\rangle_{\alpha \in \mu}$.

References


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