EACH $\mathbb{R}^\infty$-MANIFOLD HAS A UNIQUE PIECEWISE LINEAR $\mathbb{R}^\infty$-STRUCTURE

KATSURO SAKAI

Abstract. R. E. Heisey introduced piecewise linear $\mathbb{R}^\infty$-structures and defined piecewise linear $\mathbb{R}^\infty$-manifolds. In this paper we show that two piecewise linear $\mathbb{R}^\infty$-manifolds are isomorphic if they have the same homotopy type. From the Open Embedding Theorem for (topological) $\mathbb{R}^\infty$-manifolds and this result, we have the title.

Let $\mathbb{R}^\infty = \text{dir lim} \mathbb{R}^n$, the countable direct limit of real lines. We identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, so $\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$. Paracompact (topological) manifolds modeled on $\mathbb{R}^\infty$ are called $\mathbb{R}^\infty$-manifolds. Aspects of $\mathbb{R}^\infty$-manifolds have been studied by R. E. Heisey, V. T. Liem, et al. Their works show that the behavior of $\mathbb{R}^\infty$-manifolds is similar to that of $l^2$-manifolds or $Q$-manifolds (cf. references of [4]).

In [1] Heisey defined $\mathbb{R}^\infty$-piecewise linear ($\mathbb{R}^\infty$-p.l.) maps between open subsets of $\mathbb{R}^\infty$ and introduced the notion of piecewise linear $\mathbb{R}^\infty$-structure ($\text{p.l. } \mathbb{R}^\infty$-structure) as in differential topology. A piecewise linear $\mathbb{R}^\infty$-manifold ($\text{p.l. } \mathbb{R}^\infty$-manifold) is a paracompact space together with a p.l. $\mathbb{R}^\infty$-structure. Clearly p.l. $\mathbb{R}^\infty$-manifolds are (topological) $\mathbb{R}^\infty$-manifolds. And $\mathbb{R}^\infty$-p.l. maps and $\mathbb{R}^\infty$-p.l. isomorphisms between two p.l. $\mathbb{R}^\infty$-manifolds are defined similarly as differential maps and diffeomorphisms between two differential manifolds. For the precise definitions and the undefined terms used in this paper, refer to [1].

We assume separability for all manifolds throughout this paper. In this paper we show the following:

Theorem. Two p.l. $\mathbb{R}^\infty$-manifolds are $\mathbb{R}^\infty$-p.l. isomorphic if they have the same homotopy type.

By the Open Embedding Theorem [2] (cf. [4]), each $\mathbb{R}^\infty$-manifold can be embedded as an open subset of $\mathbb{R}^\infty$. Since open subsets of $\mathbb{R}^\infty$ have the p.l. $\mathbb{R}^\infty$-structure inherited from $\mathbb{R}^\infty$, we have the title’s result.

Corollary. Each $\mathbb{R}^\infty$-manifold has a unique p.l. $\mathbb{R}^\infty$-structure.

The proof of the theorem proceeds in the spirit of [4]. In order to use the general position arguments (e.g. see 5.4 in [3]) instead of [4, Lemma 1-5], we have to see that
each p.l. $\mathbb{R}^\infty$-manifold can be expressed as a direct limit of p.l. embedded (finite-di-

Each $\mathbb{R}^\infty$-manifold has a unique PL $\mathbb{R}^\infty$-structure

Lemma. Let $M$ be a closed p.l. $\mathbb{R}^\infty$-submanifold of $\mathbb{R}^\infty$, $C$ a compact subset of $M$ and $m_0$ a positive integer. Then there exists a compact polyhedral $m$-manifold $N$ in $\mathbb{R}^\infty$ with $C \subset N \subset M$, where $m \geq m_0$ and $N$ is the interior of a manifold.

We need to strengthen [1, Lemma 6] a little.

Sublemma. Any p.l. $\mathbb{R}^\infty$-atlas for $M$ contains finitely many p.l. $\mathbb{R}^\infty$-charts $(U_i, \phi_i), \ldots, (U_n, \phi_n)$ such that there are $m$-cubes $D_1, \ldots, D_n$ in $\mathbb{R}^m$, $m \geq m_0$, with $D_i \subset \phi_i(U_i)$ and $C \subset \bigcup_{i=1}^n \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i)$; moreover, $\phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i) \cap C$ is open in $C$.

Proof (Cf. Proof of [1, Lemma 6]). For each $x \in C$, take a p.l. $\mathbb{R}^\infty$-chart $(U_x, \phi_x)$ from the given atlas and an open set $V_x$ so that $x \in V_x \subset \text{cl} V_x \subset U_x$. Let $\phi(x) = (y_i) \in \mathbb{R}^\infty$ and choose $D'_i = (\prod_{i \in \mathbb{N}} [y_i - \varepsilon_i, y_i + \varepsilon_i]) \cap \mathbb{R}^\infty$ so that $D'_i \subset \phi_i(V_x)$. From compactness,

$$C \subset \bigcup_{i=1}^n \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D'_i)$$

for some $x_1, \ldots, x_n \in C$. Put $(U_i, \phi_i) = (U_{x_i}, \phi_{x_i})$, $V_i = V_{x_i}$ and $D'_i = D'_i$. From compactness, $\phi_i(C \cap \text{cl} V_i) \subset \mathbb{R}^m$ for some $m_i$. Let $m = \max\{m_0, m_1, \ldots, m_n\}$ and $D_i = D'_i \cap \mathbb{R}^m$. Then

$$\phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i) \cap C \cap V_i = \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D'_i) \cap C \cap V_i,$$

hence $\phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i) \cap C$ is open in $C$, and

$$C = \bigcup_{i=1}^n \left( \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D'_i) \cap C \cap V_i \right) \subset \bigcup_{i=1}^n \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i).$$

Proof of Lemma (Cf. Proof of [1, Proposition 7]). Using the above, we have p.l. $\mathbb{R}^\infty$-charts $(U_i, \phi_i), \ldots, (U_n, \phi_n)$ for $M$ and $m$-cubes $D_1, \ldots, D_n$ in $\mathbb{R}^m$, $m \geq m_0$, such that $D_i \subset \phi_i(U_i)$, $C \subset \bigcup_{i=1}^n \phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i)$, each $\phi_i^{-1}(\text{int}_{\mathbb{R}^\infty} D_i) \cap C$ is open in $C$ and, moreover, each $(U_i, \phi_i)$ extends to a p.l. $\mathbb{R}^\infty$-chart $(\tilde{U}_i, \tilde{\phi}_i)$ for $\mathbb{R}^\infty$, that is, $\tilde{\phi}_i: \tilde{U}_i \to \phi_i(U_i) \times V_i$, $V_i$ open in $\mathbb{R}^\infty$ and $(U_i, \phi_i) = (\tilde{U}_i \cap M, \tilde{\phi}_i | \tilde{U}_i \cap M)$. Then $P = \bigcup_{i=1}^n \phi_i^{-1}(D_i)$ is a compact polyhedron in $\mathbb{R}^\infty$ with $C \subset P \subset M$ since so is $\phi_i^{-1}(D_i) = \phi_i^{-1}(D_i \times \{0\})$. Although $P$ need not be a manifold, every point of $C$ has a neighborhood in $P$ which is an $m$-cube. Hence $P$ contains a compact polyhedral $m$-manifold $N$ which is a neighborhood of $C$ in $P$, therefore $C \subset \tilde{N} \subset M$. □

Now we will prove the following

Proposition. Any closed p.l. $\mathbb{R}^\infty$-submanifold $M$ of $\mathbb{R}^\infty$ is dir lim $M_i$, where each $M_i$ is a compact polyhedral $m_i$-manifold in $\mathbb{R}^\infty$ with $M_i \subset M_{i+1}$, $m_i < m_{i+1}$.

Proof. First write $M = \text{dir lim } X_n$, where each $X_n$ is a finite-dimensional compact metrizable subspace of $X_{n+1}$. Using the Lemma, we have compact polyhedral $m_i$-manifold $M_i$, $i = 1, 2, \ldots$, in $\mathbb{R}^\infty$ and integers $0 < n_1 < n_2 < \cdots$ with

$$X_{n_1} \subset M_1 \subset X_{n_1} \subset X_{n_2} \subset M_2 \subset \cdots; \quad m_1 < m_2 < \cdots.$$

Then clearly $M = \text{dir lim } M_i$. □
Proof of Theorem. Let $M$ and $N$ be p.l. $\mathbb{R}^\infty$-manifolds which have the same homotopy type. By [1, Theorem], we may assume $M$ and $N$ are closed p.l. $\mathbb{R}^\infty$-submanifolds of $\mathbb{R}^\infty$. From the Proposition, $M = \text{dir lim } M_i$ and $N = \text{dir lim } N_i$, where each $M_i$ and $N_i$ are compact polyhedral $m_i$- and $n_i$-manifolds in $\mathbb{R}^\infty$ with $M_i \subset M_{i+1}$, $N_i \subset N_{i+1}$, $m_i < m_{i+1}$, $n_i < n_{i+1}$. Let $f: M \to N$ be a homotopy equivalence with a homotopy inverse $g: N \to M$. Put $i_1 = 1$. From compactness we can choose a $j_1 > 1$ so that $f(M_{i_1}) \subset N_{j_1}$ and $2 \cdot m_{i_1} < n_{j_1}$. By the general position arguments (cf. 5.4 in [3]), we have a p.l. embedding $f_1: M_{i_1} \to N_{j_1}$ homotopic to $f| M_{i_1}$. Since $f^{-1}_1: f_1(M_{i_1}) \to M$ is homotopic to $g| f_1(M_{i_1})$, $f^{-1}_1$ extends to a map $g_1: N_{j_1} \to M$ homotopic to $g| N_{j_1}$. From compactness, choose an $i_2 > i_1$ so that $g_1(N_{j_1}) \subset M_{i_2}$ and $2 \cdot n_{j_1} > m_{i_2}$. Again by the general position arguments, we have a p.l. embedding $g_1: N_{j_1} \to M_{i_2}$ which extends $f_1$ and is homotopic to $g_1$, hence to $g| N_{j_1}$. Since $g^{-1}_1: g_1(N_{j_1}) \to N$ is homotopic to $f| g_1(N_{j_1})$, $g^{-1}_1$ extends to a map $f_2: M_{i_2} \to N_{i_2}$ homotopic to $f| M_{i_2}$. Similarly as above, we have a $j_2 > j_1$ and a p.l. embedding $f_2: M_{i_2} \to N_{j_2}$ which extends $g_1$ and is homotopic to $f| M_{i_2}$. Thus inductively we have the following commutative diagram of p.l. embeddings:

$$
\begin{array}{c}
\begin{array}{cccc}
M_{i_1} & \subset & M_{i_2} & \subset & \cdots \\
\downarrow f_1 & & \downarrow g_1 & & \\
N_{j_1} & \subset & N_{j_2} & \subset & \cdots
\end{array}
\end{array}
$$

Then $f_1$, $f_2$, ... induce a homeomorphism $f_\infty: M \to N$. Evidently for each compact polyhedron $P \subset M$ and any choice of $n$ so that $f_\infty(P) \subset N \cap \mathbb{R}^n$, $f_\infty| P: P \to N \cap \mathbb{R}^n$ is p.l., then $f_\infty$ is $\mathbb{R}^\infty$-p.l. in the polyhedral sense, so $\mathbb{R}^\infty$-p.l. in the manifold sense by [1, Proposition 8]. From [1, Proposition 5], $f_\infty$ is an $\mathbb{R}^\infty$-p.l. isomorphism. This completes the proof. \(\square\)

Remark. Using arguments in the proof of [4, Theorem 2-3], we can prove that every fine homotopy equivalence between p.l. $\mathbb{R}^\infty$-manifolds can be approximated by $\mathbb{R}^\infty$-p.l. isomorphisms. Especially, every homeomorphism between p.l. $\mathbb{R}^\infty$-manifolds is homotopic to an $\mathbb{R}^\infty$-p.l. isomorphism by a small homotopy. By combining [4, Theorem 2-2] and this result, we can assert that any map $f: M \to N$ between p.l. $\mathbb{R}^\infty$-manifolds can be approximated by p.l. isomorphisms onto open submanifolds of $N$.

References


Institute of Mathematics, University of Tsukuba, Sakuramura, Ibaraki 305, Japan