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EACH $\mathbf{R}^\infty$-MANIFOLD HAS A UNIQUE PIECEWISE LINEAR $\mathbf{R}^\infty$-STRUCTURE

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ABSTRACT. R. E. Heisey introduced piecewise linear $\mathbf{R}^\infty$-structures and defined piecewise linear $\mathbf{R}^\infty$-manifolds. In this paper we show that two piecewise linear $\mathbf{R}^\infty$-manifolds are isomorphic if they have the same homotopy type. From the Open Embedding Theorem for (topological) $\mathbf{R}^\infty$-manifolds and this result, we have the title.

Let $\mathbf{R}^\infty = \operatorname{dir}\lim \mathbf{R}^n$, the countable direct limit of real lines. We identify $\mathbf{R}^n$ with $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$, so $\mathbf{R}^\infty = \bigcup_{n\in\mathbf{N}} \mathbf{R}^n$. Paracompact (topological) manifolds modeled on $\mathbf{R}^\infty$ are called *$\mathbf{R}^\infty$-manifolds*. Aspects of $\mathbf{R}^\infty$-manifolds have been studied by R. E. Heisey, V. T. Liem, et al. Their works show that the behavior of $\mathbf{R}^\infty$-manifolds is similar to that of l^2-manifolds or Q-manifolds (cf. references of [**4**]).

In [**1**] Heisey defined *$\mathbf{R}^\infty$-piecewise linear* (*$\mathbf{R}^\infty$-p.l.*) *maps* between open subsets of $\mathbf{R}^\infty$ and introduced the notion of *piecewise linear $\mathbf{R}^\infty$-structure* (*p.l. $\mathbf{R}^\infty$-structure*) as in differential topology. A *piecewise linear $\mathbf{R}^\infty$-manifold* (*p.l. $\mathbf{R}^\infty$-manifold*) is a paracompact space together with a p.l. $\mathbf{R}^\infty$-structure. Clearly p.l. $\mathbf{R}^\infty$-manifolds are (topological) $\mathbf{R}^\infty$-manifolds. And *$\mathbf{R}^\infty$-p.l. maps* and *$\mathbf{R}^\infty$-p.l. isomorphisms* between two p.l. $\mathbf{R}^\infty$-manifolds are defined similarly as differential maps and diffeomorphisms between two differential manifolds. For the precise definitions and the undefined terms used in this paper, refer to [**1**].

We assume *separability* for all manifolds throughout this paper. In this paper we show the following:

THEOREM. *Two p.l. $\mathbf{R}^\infty$-manifolds are $\mathbf{R}^\infty$-p.l. isomorphic if they have the same homotopy type.*

By the Open Embedding Theorem [**2**] (cf. [**4**]), each $\mathbf{R}^\infty$-manifold can be embedded as an open subset of $\mathbf{R}^\infty$. Since open subsets of $\mathbf{R}^\infty$ have the p.l. $\mathbf{R}^\infty$-structure inherited from $\mathbf{R}^\infty$, we have the title's result.

COROLLARY. *Each $\mathbf{R}^\infty$-manifold has a unique p.l. $\mathbf{R}^\infty$-structure.*

The proof of the theorem proceeds in the spirit of [**4**]. In order to use the general position arguments (e.g. see 5.4 in [**3**]) instead of [**4**, Lemma 1-5], we have to see that

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each p.l. $\mathbf{R}^\infty$-manifold can be expressed as a direct limit of p.l. embedded (finite-dimensional) compact p.l. manifolds. First we will prove the following lemma which is essentially due to Heisey [**1**].

LEMMA. *Let M be a closed p.l. $\mathbf{R}^\infty$-submanifold of $\mathbf{R}^\infty$, C a compact subset of M and m_0 a positive integer. Then there exists a compact polyhedral m-manifold N in $\mathbf{R}^\infty$ with $C \subset \mathring{N} \subset M$, where $m \geqslant m_0$ and $\mathring{N}$ is the interior of a manifold.*

We need to strengthen [**1**, Lemma 6] a little.

SUBLEMMA. *Any p.l. $\mathbf{R}^\infty$-atlas for M contains finitely many p.l. $\mathbf{R}^\infty$-charts $(U_1, \phi_1), \dots, (U_n, \phi_n)$ such that there are m-cubes $D_1, \dots, D_n$ in $\mathbf{R}^m$, $m \geqslant m_0$, with $D_i \subset \phi_i(U_i)$ and $C \subset \bigcup_{i=1}^n \phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i)$; moreover, $\phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i) \cap C$ is open in C.*

PROOF (Cf. Proof of [**1**, LEMMA 6]). For each $x \in C$, take a p.l. $\mathbf{R}^\infty$-chart (U_x, ϕ_x) from the given atlas and an open set V_x so that $x \in V_x \subset \operatorname{cl} V_x \subset U_x$. Let $\phi(x) = (y_i) \in \mathbf{R}^\infty$ and choose $D'_x = (\prod_{i \in \mathbf{N}}[y_i - \varepsilon_i, y_i + \varepsilon_i]) \cap \mathbf{R}^\infty$ so that $D'_x \subset \phi_x(V_x)$. From compactness,

$$C \subset \bigcup_{i=1}^{n} \phi_{x_i}^{-1}\left(\operatorname{int}_{\mathbf{R}^\infty} D'_{x_i}\right) \quad \text{for some } x_1, \dots, x_n \in C.$$

Put $(U_i, \phi_i) = (U_{x_i}, \phi_{x_i})$, $V_i = V_{x_i}$ and $D'_i = D'_{x_i}$. From compactness, $\phi_i(C \cap \operatorname{cl} V_i) \subset \mathbf{R}^{m_i}$ for some m_i. Let $m = \max\{m_0, m_1, \dots, m_n\}$ and $D_i = D'_i \cap \mathbf{R}^m$. Then

$$\phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i) \cap C \cap V_i = \phi_i^{-1}(\operatorname{int}_{\mathbf{R}^\infty} D'_i) \cap C \cap V_i,$$

hence $\phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i) \cap C$ is open in C, and

$$C = \bigcup_{i=1}^{n} \left(\phi_i^{-1}(\operatorname{int}_{\mathbf{R}^\infty} D'_i) \cap C \cap V_i\right) \subset \bigcup_{i=1}^{n} \phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i). \quad \square$$

PROOF OF LEMMA (Cf. Proof of [**1**, PROPOSITION 7]). Using the above, we have p.l. $\mathbf{R}^\infty$-charts $(U_1, \phi_1), \dots, (U_n, \phi_n)$ for M and m-cubes $D_1, \dots, D_n$ in $\mathbf{R}^m$, $m \geqslant m_0$, such that $D_i \subset \phi_i(U_i)$, $C \subset \bigcup_{i=1}^n \phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i)$, each $\phi_i^{-1}(\operatorname{int}_{\mathbf{R}^m} D_i) \cap C$ is open in C and, moreover, each (U_i, ϕ_i) extends to a p.l. $\mathbf{R}^\infty$-chart $(\tilde{U}_i, \tilde{\phi}_i)$ for $\mathbf{R}^\infty$, that is, $\tilde{\phi}_i$: $\tilde{U}_i \to \phi_i(U_i) \times V_i$, V_i open in $\mathbf{R}^\infty$ and $(U_i, \phi_i) = (\tilde{U}_i \cap M, \tilde{\phi}_i | \tilde{U}_i \cap M)$. Then $P = \bigcup_{i=1}^n \phi_i^{-1}(D_i)$ is a compact polyhedron in $\mathbf{R}^\infty$ with $C \subset P \subset M$ since so is $\phi_i^{-1}(D_i) = \tilde{\phi}_i^{-1}(D_i \times \{0\})$. Although P need not be a manifold, every point of C has a neighborhood in P which is an m-cube. Hence P contains a compact polyhedral m-manifold N which is a neighborhood of C in P, therefore $C \subset \mathring{N} \subset M$. $\square$

Now we will prove the following

PROPOSITION. *Any closed p.l. $\mathbf{R}^\infty$-submanifold M of $\mathbf{R}^\infty$ is* dir lim *M_i where each M_i is a compact polyhedral m_i-manifold in $\mathbf{R}^\infty$ with $M_i \subset \mathring{M}_{i+1}$, $m_i < m_{i+1}$.*

PROOF. First write $M = \operatorname{dir\,lim} X_n$, where each X_n is a finite-dimensional compact metrizable subspace of X_{n+1}. Using the Lemma, we have compact polyhedral m_i-manifold M_i, $i = 1, 2, \dots$, in $\mathbf{R}^\infty$ and integers $0 < n_1 < n_2 < \cdots$ with

$$X_{n_1} \subset \mathring{M}_1 \subset M_1 \subset X_{n_2} \subset \mathring{M}_2 \subset M_2 \subset \cdots; \quad m_1 < m_2 < \cdots.$$

Then clearly $M = \operatorname{dir\,lim} M_i$. $\square$

Proof of Theorem. Let M and N be p.l. $\mathbf{R}^\infty$-manifolds which have the same homotopy type. By [**1**, Theorem], we may assume M and N are closed p.l. $\mathbf{R}^\infty$-submanifolds of $\mathbf{R}^\infty$. From the Proposition, $M = \operatorname{dir\,lim} M_i$ and $N = \operatorname{dir\,lim} N_i$ where each M_i and N_i are compact polyhedral m_i- and n_i-manifolds in $\mathbf{R}^\infty$ with $M_i \subset \mathring{M}_{i+1}$, $N_i \subset \mathring{N}_{i+1}$; $m_i < m_{i+1}$, $n_i < n_{i+1}$. Let $f\colon M \to N$ be a homotopy equivalence with a homotopy inverse $g\colon N \to M$. Put $i_1 = 1$. From compactness we can choose a $j_1 > 1$ so that $f(M_{i_1}) \subset \mathring{N}_{j_1}$ and $2 \cdot m_{i_1} < n_{j_1}$. By the general position arguments (cf. 5.4 in [3]), we have a p.l. embedding $f_1\colon M_{i_1} \to N_{j_1}$ homotopic to $f | M_{i_1}$. Since $f_1^{-1}\colon f_1(M_{i_1}) \to M$ is homotopic to $g | f_1(M_{i_1})$, f_1^{-1} extends to a map $g_1'\colon N_{j_1} \to M$ homotopic to $g | N_{j_1}$. From compactness, choose an $i_2 > i_1$ so that $g_1'(N_{j_1}) \subset \mathring{M}_{i_2}$ and $2 \cdot n_{j_1} > m_{i_2}$. Again by the general position arguments, we have a p.l. embedding $g_1\colon N_{j_1} \to M_{i_2}$ which extends f_1^{-1} and is homotopic to g_1', hence to $g | N_{j_1}$. Since $g_1^{-1}\colon g_1(N_{j_1}) \to N$ is homotopic to $f | g_1(N_{j_1})$, g_1^{-1} extends to a map $f_2'\colon M_{i_2} \to N$ homotopic to $f | M_{i_2}$. Similarly as above, we have a $j_2 > j_1$ and a p.l. embedding $f_2\colon M_{i_2} \to N_{j_2}$ which extends g_1^{-1} and is homotopic to $f | M_{i_2}$. Thus inductively we have the following commutative diagram of p.l. embeddings:

$$\begin{tikzcd}
M_{i_1} \arrow[d, "f_1"'] & \subset & M_{i_2} \arrow[d, "f_2"'] & \subset & M_{i_3} \arrow[d, "f_3"'] & \subset & \cdots \\
N_{j_1} \arrow[urr, "g_1"] & \subset & N_{j_2} \arrow[urr, "g_2"] & \subset & N_{j_3} \arrow[urr, "g_3"] & \subset & \cdots
\end{tikzcd}$$

Then $f_1, f_2, \ldots$ induce a homeomorphism $f_\infty\colon M \to N$. Evidently for each compact polyhedron $P \subset M$ and any choice of n so that $f_\infty(P) \subset N \cap \mathbf{R}^n$, $f_\infty | P\colon P \to N \cap \mathbf{R}^n$ is p.l., then f_∞ is $\mathbf{R}^\infty$-p.l. in the polyhedral sense, so $\mathbf{R}^\infty$-p.l. in the manifold sense by [**1**, Proposition 8]. From [**1**, Proposition 5], f_∞ is an $\mathbf{R}^\infty$-p.l. isomorphism. This completes the proof. □

Remark. Using arguments in the proof of [**4**, Theorem 2-3], we can prove that *every fine homotopy equivalence between p.l.* $\mathbf{R}^\infty$*-manifolds can be approximated by* $\mathbf{R}^\infty$*-p.l. isomorphisms*. Especially, *every homeomorphism between p.l.* $\mathbf{R}^\infty$*-manifolds is homotopic to an* $\mathbf{R}^\infty$*-p.l. isomorphism by a small homotopy*. By combining [**4**, Theorem 2-2] and this result, we can assert that *any map* $f\colon M \to N$ *between p.l.* $\mathbf{R}^\infty$*-manifolds can be approximated by p.l. isomorphisms onto open submanifolds of* N.

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