A REMARK ON REFINABLE MAPS AND CALMNESS

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Abstract. It is shown that if $r : X \to Y$ is a refinable map between compacta and $Y$ is calm, then $r$ is a shape equivalence. As a corollary, if $r : X \to Y$ is a refinable map between compacta and either $X$ or $Y$ is $S^n$-like ($n \geq 1$), then $r$ is a shape equivalence, where $S^n$ denotes the $n$-sphere.

0. Introduction. In [9], J. Ford and J. W. Rogers, Jr. introduced the notion of refinable maps and they proved several results about these maps. In [10], we showed that every refinable map does not preserve shape (cf. [12]) and introduced the notion of pseudo-isomorphisms in pro-category, and by using this notion we proved that if $r : X \to Y$ is a refinable map between compacta and $Y$ is an FANR, then $r$ is a shape equivalence. In this paper, we will give the following more general result: if $r : X \to Y$ is a refinable map between compacta and $Y$ is calm, then $r$ is a shape equivalence. The notion of calmness was introduced by Z. Čerin [4]. It is well known that the dyadic solenoid is calm but not movable [4]. As a corollary, we obtain that if $r : X \to Y$ is a refinable map between compacta and either $X$ or $Y$ is $S^n$-like ($n \geq 1$), then $r$ is a shape equivalence. In relation to the above result, the following are known; if $r : X \to Y$ is a refinable map between compacta, then $r$ is a weakly confluent map [9], moreover;

1) if $Y$ has property $[K]$ due to J. L. Kelley [15, (3.2)], then $r$ is a confluent map [14],

2) if $Y$ is locally connected, then $r$ is a monotone map [9],

3) if $Y$ is locally $n$-connected ($n \geq 1$), then $r$ is a $UV^n$-map [11], and

4) if $Y$ is an ANR (locally contractible), then $r$ is a cell-like map [11].

Several properties concerning the notions of refinable maps, ARI-maps, AI-maps, calmness, AANR and quasi-ANR etc. have been studied in [1, 3–14, 16, 17 etc.].

1. Definitions and notations. For a metric space $X$, if $x$ and $y$ are points of $X$, $d(x, y)$ denotes the distance from $x$ to $y$. A map $r : X \to Y$ between compacta is refinable [9] if for any $\epsilon > 0$ there is a surjective map $f : X \to Y$ such that $\text{diam} f^{-1}(y) < \epsilon$ for each $y \in Y$ and $d(r, f) = \sup \{d(r(x), f(x)) \mid x \in X\} < \epsilon$. Note that every refinable map is surjective, every near-homeomorphism is refinable and if there is a refinable map from a compactum $X$ to a compactum $Y$, then $X$ is $Y$-like. But simple examples show that the converse of any of these assertions is not true. A compactum $X$ is calm [4] if whenever $X \subset M \subset \text{ANR}$, there is a neighborhood $V$ of $X$ in $M$ such that for any neighborhood $U$ of $X$ in $M$ there is a neighborhood $W$ of $X$ in $M$, $W \subset U$, such that if $f, g : Y \to W$ are maps with $f \simeq g$ in $V$, then $f \simeq g$ in $U$ for all $Y \in \text{ANR}$. Let $K$ be an arbitrary category. By $K$, we mean the category of inverse systems in $K$ and system maps in $K$, also by pro-$K$, the

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homotopy category of K. An inverse system \( \{X_\alpha, p_{\alpha\alpha}, A\} \) of pro-\( K \) is calm if there is \( \alpha_0 \in A \) such that

for any \( \alpha \in A (\alpha \geq \alpha_0) \) there is \( \beta \in A, \beta \geq \alpha, \) such that

\[
(*) \quad \text{if } f, g : Y \rightarrow X_\beta \text{ are morphisms in } K \text{ with } p_{\alpha\alpha} f = p_{\alpha\beta} g,
\]

then \( p_{\alpha\beta} f = p_{\alpha\beta} g \) for all \( Y \in K \).

It is easily seen that a compactum \( X \) is calm iff there is an inverse system \( \{X_n, [p_{nn+1}]\} \) in pro-HCW which is calm and associated with \( X \), where HCW is the category of spaces having the homotopy type of CW-complexes and homotopy classes of maps. A system map \( f = \{f, f_\beta, B\} : \{X_\alpha, p_{\alpha\alpha}, A\} \rightarrow \{Y_\beta, q_{\beta\beta}, B\} \) of \( K \) is a pseudo-isomorphism [10] if for each \( \beta \in B \) and each \( \alpha \geq f(\beta) \) there exist \( g(\alpha, \beta) \geq \beta \) and a morphism \( g(\alpha, \beta) : Y_\gamma(\alpha, \beta) \rightarrow X_\alpha \) such that

\[
(**) \quad \text{for every } \beta' \geq g(\alpha, \beta) \text{ there exist } h(\beta') \geq \alpha \text{ and a morphism } h_{\beta'} : X_{h(\beta')} \rightarrow Y_{\beta'} \text{ such that}
\]

\[
f_{\beta' P_{f(\beta')}} g(\alpha, \beta) = g_{\beta' g(\alpha, \beta)} \quad \text{and} \quad g_{(\alpha, \beta)} q_{g(\alpha, \beta)\beta'} h_{\beta'} = p_{\alpha h(\beta')}.
\]

A morphism \( f : X = \{X_\alpha, p_{\alpha\alpha}, A\} \rightarrow Y = \{Y_\beta, q_{\beta\beta}, B\} \) of pro-\( K \) is a pseudo-isomorphism if it has a pseudo-isomorphism \( f : X \rightarrow Y \) of \( K \) as the representation, i.e. \( f = [f] \).

**Theorem (1.1)** [7]. A compactum \( X \) is an FANR iff \( X \) is calm and movable.

**Theorem (1.2)** [1]. A compactum \( X \) is an AANR\(_N\) iff \( X \) is an AANR\(_C\) and an FANR.

2. Calmness and pseudo-isomorphisms. In this section, we show that if \( r : X \rightarrow Y \) is a refinable map between compacta and \( Y \) is calm, then \( r \) is a shape equivalence. First, we prove the following lemma (cf. [10, (2.1)]).

**Lemma (2.1).** For a category \( K \), if \( f : X = \{X_\alpha, p_{\alpha\alpha}, A\} \rightarrow Y = \{Y_\beta, q_{\beta\beta}, B\} \) is a pseudo-isomorphism in pro-\( K \) and \( Y \) is calm, then \( f \) is an isomorphism in pro-\( K \).

**Proof.** Let \( f = \{f, f_\beta, B\} : X \rightarrow Y \) be a pseudo-isomorphism of \( K \) such that \( f = [f] \). Since \( Y \) is calm, there is \( \beta_0 \in B \) satisfying the condition (\(*\)). Let \( \beta \geq \beta_0 \). Since \( f \) is a pseudo-isomorphism, for each \( \alpha \geq f(\beta) \) there exist \( g(\alpha, \beta) \geq \beta \) and a morphism \( g(\alpha, \beta) : Y_{g(\alpha, \beta)} \rightarrow X_\alpha \) such that

\[
(1) \quad f_{\beta P_{f(\beta)}} g(\alpha, \beta) = q_{\beta g(\alpha, \beta)}
\]

and the condition (\(**\)) is satisfied. By the choice of \( \beta_0 \), we can choose \( \beta' \geq g(\alpha, \beta) \) such that if \( f, g : Y \rightarrow Y_{\beta'} \) are morphisms in \( K \) with \( q_{\beta_0\beta} f = q_{\beta_0\beta} g \), then \( q_{g(\alpha, \beta)\beta'} f = q_{g(\alpha, \beta)\beta'} g \) for all \( Y \in K \). Also, by the condition (\(**\)), there exist \( h(\beta') \geq f(\beta') \) and a morphism \( h_{\beta'} : X_{h(\beta')} \rightarrow Y_{\beta'} \) such that

\[
(2) \quad g(\alpha, \beta) q_{g(\alpha, \beta)\beta'} h_{\beta'} = p_{\alpha h(\beta')}
\]

and

\[
(3) \quad f_{\beta P_{f(\beta)}} h(\beta') = q_{\beta\beta'} f_{\beta P_{f(\beta)}} h(\beta').
\]
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Then by (1)–(3), we have

\[ q_{\alpha\beta\gamma}(\alpha,\beta)q_{\beta}(\alpha,\beta)h_{\beta'} = q_{\alpha\beta\gamma}f_{\beta}P_{\beta}(\alpha,\beta)q_{\beta}(\alpha,\beta)h_{\beta'} = q_{\alpha\beta\gamma}f_{\beta}P_{\beta}(\alpha,\beta)h_{\beta'} \]

(4)

By the condition (*), we have

\[ q_{\alpha\beta\gamma}(\alpha,\beta)q_{\beta}(\alpha,\beta)h_{\beta'} = q_{\alpha\beta\gamma}f_{\beta}P_{\beta}(\alpha,\beta)h_{\beta'} \]

(5)

Hence by (2) and (5), we have

\[ g_{\alpha\beta\gamma}(\alpha,\beta)h_{\beta'} = q_{\alpha\beta\gamma}f_{\beta}P_{\beta}(\alpha,\beta)h_{\beta'} \]

By (1) and (6), we can conclude that the morphism \( f : X \to Y \) is an isomorphism in pro-\( K \).

**Theorem (2.2)** [10, 1.5 Theorem]. If \( r : X \to Y \) is a refinable map between compacta, then \( r \) induces a pseudo-isomorphism in pro-HCW.

By using (2.1) and (2.2), we obtain the following theorem (cf. [3, 3.3 Corollary]).

**Theorem (2.3).** If \( r : X \to Y \) is a refinable map between compacta and \( Y \) is calm, then \( r \) is a shape equivalence, i.e., \( \text{Sh}(X) = \text{Sh}(Y) \).

Combining Theorems (1.1), (1.2) and (2.3), we have

**Corollary (2.4)** [10, 2.2 Theorem]. If \( r : X \to Y \) is a refinable map between compacta and \( Y \) is an FANR, then \( r \) is a shape equivalence.

**Corollary (2.5).** If \( r : X \to Y \) is a refinable map between compacta and \( Y \) is an AANR, then \( r \) is a shape equivalence.

**Remark (2.6).** In the statement of (2.3), we cannot replace “calm” by “movable”, also in (2.5), we cannot replace AANR\( \_N \) by AANR\( \_C \) (see [10, Examples (2.5), (2.6) and (2.8)]).

Let \( \mathfrak{P} \) be a class of ANR-sets. A compactum \( X \) is \( \mathfrak{P} \)-calm [4] if whenever \( X \subset M \subset \text{ANR} \), there is a neighborhood \( V \) of \( X \) in \( M \) such that for every neighborhood \( U \) of \( X \) in \( M \) there is a neighborhood \( W, W \subset U \), such that if \( f, g : Y \to W \) with \( f \approx g \) in \( V \), then \( f \approx g \) in \( U \) for all \( Y \in \mathfrak{P} \). By using this notion, we obtain a more general result than (2.3) as follows.

**Theorem (2.7).** If \( r : X \to Y \) is a refinable map between compacta and \( X \) is \( \mathfrak{P} \)-like and \( Y \) is \( \mathfrak{P} \)-calm, then \( r \) is a shape equivalence.

**Proof.** Observe the proofs of [10, 1.5 Theorem] and (2.3).

**Corollary (2.8).** If \( r : X \to Y \) is a refinable map between compacta and if either \( X \) or \( Y \) is \( S^n \)-like \((n \geq 1)\), then \( r \) is a shape equivalence, where \( S^n \) denotes the \( n \)-sphere.
Proof. By [9, Corollary 3.1], we conclude that $X$ and $Y$ are $S^n$-like. Let 
$\{Y_i, p_{i+1}\}$ be an inverse sequence such that each $Y_i$ is the $n$-sphere ($= S^n$) and $Y = \lim\inf\{Y_i, p_{i+1}\}$. Note that $\pi_n(p_{i+1}) : \pi_n(Y_{i+1}) \rightarrow \pi_n(Y_i)$ is either a monomorphism or a zero homomorphism for each $i = 1, 2, \ldots$. Hence we can conclude that $Y$ is $S^n$-calm. Theorem (2.8) implies that $r$ is a shape equivalence.

We finish this paper with some open questions.

**Question 1.** Does every refinable map preserve calmness?

**Question 2.** Does every refinable map preserve FANR? (For the partial positive answer, see [10, Theorems 2.4, 2.12].)

**Question 3.** Does every refinable map preserve AANR? The positive answer of Question 1 implies the positive answer of Question 2 and the positive answer of Question 2 implies the positive answer of Question 3.

References


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