ABELIAN $p$-GROUPS $A$ AND $B$ SUCH THAT
$\text{Tor}(A, G) \cong \text{Tor}(B, G)$, $G$ REDUCED

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Abstract. Let $A$ be an abelian $p$-group having all of its finite Ulm invariants nonzero. Let $C$ be a countable direct sum of cyclic $p$-groups such that for each nonnegative integer $n$, the $n$th Ulm invariant of $C$ is zero if the $n$th Ulm invariant of $A$ is finite. Then for all reduced abelian groups $G$, $\text{Tor}(G, A) \cong \text{Tor}(G, A \oplus C)$.

In [2], we gave an example of two nonisomorphic $p^{\omega+1}$-projective abelian $p$-groups $A$ and $A'$ with the property that for all reduced abelian groups $G$, $\text{Tor}(G, A) \cong \text{Tor}(G, A')$. In the example, $A' = A \oplus C$ where $C$ was a countable unbounded direct sum of cyclic $p$-groups. In this note we will show that the same techniques provide a much more general result leading to many examples of nonisomorphic abelian $p$-groups $A$ and $A'$ such that $\text{Tor}(G, A) \cong \text{Tor}(G, A')$ for all reduced abelian groups $G$. For example, if $B = \bigoplus_{\omega} \mathbb{Z}(p^{n_\omega})$ then the torsion completion, $\overline{B}$, of $B$ has no unbounded summand which is a direct sum of cyclic groups. Thus, if $C$ is a countable unbounded direct sum of cyclic groups, then $\overline{B} \not\cong C \oplus \overline{B}$. We will show that $\text{Tor}(G, \overline{B}) \cong \text{Tor}(G, \overline{B} \oplus C)$ for all reduced abelian groups $G$.

In the following, all groups will be abelian groups, $p$ is a fixed but arbitrary prime, $\mathbb{Z}(p^\omega)$ is a cyclic group of order $p^\omega$, $\omega$ is the first infinite ordinal and $\kappa$ is an infinite cardinal. If $A$ is a $p$-group then by $r(A)$ we shall mean the rank of $A$. The final rank of $A$, denoted $\text{fin} \ r(A)$, is the inf $n \in \omega \ r(p^n A)$. The notation and terminology will be the same as that in [3].

We will need the following technical lemma in the proof of our Theorem. Recall that for a $p$-group $A$ and $i \in \omega$, the $i$th Ulm invariant of $A$ not zero means that $A$ has a cyclic summand of order $p^{i+1}$.

**Lemma.** Let $0 \to B \to G \to \bigoplus_\kappa \mathbb{Z}(p^\omega) \to 0$ be a pure exact sequence where $\kappa$ is an infinite cardinal. Let $A$ be a $p$-group and $(n_i)_{i \in \omega}$ be a subsequence of $\omega$ such that the $n_i$th Ulm invariant of $A$ is not zero. Then $\text{Tor}(G, A)$ has a summand $S$ such that $S = \bigoplus_{i \in \omega} \bigoplus_\kappa \mathbb{Z}(p^{n_i+1})$.

**Proof.** By Theorem 63.2 in [3], if $0 \to B \to G \to \bigoplus_\kappa \mathbb{Z}(p^\omega) \to 0$ is pure exact then the induced sequence $0 \to \text{Tor}(B, A) \to \text{Tor}(G, A) \to \bigoplus_\kappa A \to 0$ is pure exact for all $p$-groups $A$. Let $A$ be the induced homomorphism from $\text{Tor}(G, A)$ to $\bigoplus_\kappa A$. For each $i \in \omega$, since the $n_i$th Ulm invariant of $A$ is not zero, we have a decomposition $A = \mathbb{Z}(p^{n_i+1}) \oplus A_i$. By partitioning $\kappa$ into $N_0$ sets of cardinality $\kappa$,

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we have $\bigoplus_{\kappa} A = S' \oplus A'$ where

$$S' = \bigoplus_{i \in \omega} \left( \bigoplus_{\kappa} Z(p^{n_i+1}) \right) \quad \text{and} \quad A' = \bigoplus_{i \in \omega} \left( \bigoplus_{\kappa} A_i \right).$$

Since $\text{Tor}(B,A)$ is pure in $\text{Tor}(G,A)$, we also have $\text{Tor}(B,A)$ pure in $\lambda^{-1}(S')$. By this purity and the fact that $S'$ is a direct sum of cyclic groups, it follows from Theorem 28.2 of [3] that $\lambda^{-1}(S') = \text{Tor}(B,A) \oplus S$ with $S \cong S'$. Since $S \cap \lambda^{-1}(A') = 0$ and $S + \lambda^{-1}(A') = \text{Tor}(G,A)$, we have $\text{Tor}(G,A) \cong S \oplus \lambda^{-1}(A')$.

**Theorem.** Let $A$ be an abelian $p$-group such that, for all $i \in \omega$, the $i$th Ulm invariant of $A$ is not zero. Let $C$ be a countable direct sum of cyclic $p$-groups such that for all $i \in \omega$, the $i$th Ulm invariant of $C$ is zero if the $i$th Ulm invariant of $A$ is finite. Then for all reduced abelian groups $G$, $\text{Tor}(G,A) \cong \text{Tor}(G,A \oplus C)$.

**Proof.** Since $A$ and $K$ are $p$-groups, we need only consider the case in which $G$ is a $p$-group. Note that if $B$ is a basic subgroup of $A$, the condition on the Ulm invariants of $C$ and the fact that $C$ is countable implies that $B \cong B \oplus C$. It follows easily that, for all positive integers $n$, $A[p^n] \cong (A \oplus C)[p^n]$. Since $\text{Tor}(Z(p^n), A) \cong A[p^n]$ for all groups $A$, we have $\text{Tor}(Z(p^n), A) \cong \text{Tor}(Z(p^n), A \oplus C)$. Since $\text{Tor}$ commutes with direct sums, $\text{Tor}(K,A) \cong \text{Tor}(K,A \oplus C)$ for all $K$ which are direct sums of cyclic groups.

Let $G$ be an unbounded reduced $p$-group and let $B$ be a basic subgroup of $G$ such that $r(G/B) = \text{fin } r(G)$. If $r(G/B) < r(B)$, then, by Lemma 1 of [1], there is a decomposition of $G$, say $G = H \oplus L$, such that $L$ is a subgroup of $B$, $H \cap B$ is a basic subgroup of $H$, and $r(H/(H \cap B)) \geq r(H \cap B)$. Since $\text{Tor}$ commutes with direct sums and $\text{Tor}(L,A) \cong \text{Tor}(L',A')$, we need only consider unbounded reduced $p$-groups $G$ which have a basic subgroup $B$ with $r(B) \leq r(G/B)$. With this assumption, we have the pure exact sequence $0 \to B \to G \to \bigoplus_{\kappa} Z(p^\kappa) \to 0$ where $|G| = \kappa \geq |B|$. By the above Lemma, $\text{Tor}(G,A)$ has a summand $S = \bigoplus_{\kappa \in \omega} \left( \bigoplus_{\kappa} Z(p^{\kappa+1}) \right)$. Note that since $C$ is a countable direct sum of cyclic $p$-groups, $\text{Tor}(G,Z(p^n)) \cong G[p^n]$, and $\text{Tor}$ commutes with direct sums,

$$\text{Tor}(G,C) = \bigoplus_{\kappa} \left( \bigoplus_{\kappa} Z(p^{\kappa+1}) \right) \quad \text{where } \kappa_i \leq \kappa.$$

Thus $\text{Tor}(G,C) \oplus S \cong S$. Hence $\text{Tor}(G,A \oplus C) \cong \text{Tor}(G,A)$.

**Corollary.** Let $A$ be an abelian $p$-group with the following properties:

1. For $i \in \omega$, the $i$th Ulm invariant of $A$ is infinite;
2. $A$ has no unbounded summand which is a direct sum of cyclic groups.

Then for all countable unbounded direct sums of cyclic groups $C$, $A \neq A \oplus C$, but $\text{Tor}(G,A) \cong \text{Tor}(G,A \oplus C)$ for all reduced abelian groups $G$.

This leads one naturally to pose the following open problem.

**Problem.** Find an example of two abelian $p$-groups $A$ and $A'$ such that for all direct sums of cyclic groups $K$ and $K'$, $A \oplus K \neq A' \oplus K'$ but $\text{Tor}(G,A) \cong \text{Tor}(G,A')$ for all reduced abelian groups $G$.

One might note that no such example exists for the class of $p^{\omega+n}$-projective $p$-groups. To see this, suppose that $A$ and $A'$ are $p^{\omega+n}$-projective $p$-groups such that $\text{Tor}(G,A) \cong \text{Tor}(G,A')$ for all reduced abelian groups $G$. Let $C = \bigoplus_{\kappa} Z(p^{\kappa+1})$.
and let $H_n$ be a $p$-group such that $H_n/p\omega H_n \cong C$ and $p\omega H_n \cong Z(p^n)$. Let $B (\cong C)$ be a $p^{\omega+n-1}$-high subgroup of $H_n$. By Proposition 1 in [4], $B$ is $p^{\omega+n}$-pure in $H_n$. Thus the exact sequence $0 \to B \to H_n \to Z(p^{\infty}) \to 0$ is $p^{\omega+n}$-pure exact. Thus by Proposition 2 in [4], the induced sequence $0 \to \text{Tor}(A, B) \to \text{Tor}(A, H_n) \to A \to 0$ is $p^{\omega+n}$-pure exact. Thus, since $A$ is $p^{\omega+n}$-projective, $\text{Tor}(A, H_n) \cong A \oplus \text{Tor}(A, B)$. Since $\text{Tor}(A, B)$ is a direct sum of cyclic groups and $\text{Tor}(A, H_n) \cong \text{Tor}(A', H_n)$, there exist direct sums of cyclic $p$-groups $K$ and $K'$ such that $A \oplus K \cong A' \oplus K'$.

References


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