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A FORMULA FOR RAMANUJAN'S TAU FUNCTION

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ABSTRACT. A formula for Ramanujan's tau function τ, defined by $\sum_1^\infty \tau(n)x^n = x\prod_1^\infty(1-x^n)^{24}$ $(|x|<1)$, is presented. The author then observes that some of the known congruence properties of τ are immediate consequences of this formula representation.

1. Introduction. The subject of this investigation is the arithmetical function τ defined by the identity

$$\sum_1^\infty \tau(n)x^n = x\prod_1^\infty(1-x^n)^{24}, \tag{1}$$

valid for each complex number x such that $|x|<1$. As intimated in the above title, S. Ramanujan [**2**, p. 151] was the first mathematician to consider this function. We here propose to derive a formula which expresses τ in terms of two simpler arithmetical functions. Accordingly, we define these functions, as follows:

DEFINITION. Let k denote an arbitrary positive integer. Then, for each positive integer n: (i) $\sigma_k(n)$ denotes the sum of the kth powers of all positive divisors of n (of course, $\sigma_1 = \sigma$, the ordinary sum-of-divisors function); (ii) $b(n)$ denotes the exponent of the highest power of 2 dividing n; and $O(n)$ is then defined by the equation $n = 2^{b(n)}O(n)$. (Hence, $b(n)$ is a nonnegative integer and $O(n)$ is odd.) Finally, (iii) for each nonnegative integer n, $r_k(n)$ denotes the cardinal number of the set

$$\{(x_1, x_2, \ldots, x_k) \in \mathbf{Z}^k \mid n = x_1^2 + x_2^2 + \cdots + x_k^2\}.$$

(Note that $r_k(0) = 1$.)

We are now prepared to state our main result.

THEOREM 1. *For each positive integer n,*

$$\tau(n) = \sum_{i=1}^n (-1)^{n-i} r_{16}(n-i)2^{3b(i)}\sigma_3(O(i)). \tag{2}$$

The two simpler functions are then σ_3 and r_{16}. (The functions b and O are trivially defined by the fundamental theorem of arithmetic.) In §2 we prove this theorem. §3 is devoted to a discussion of some of the known congruence properties of τ. As a matter of fact, we restrict attention to those properties which are immediate consequences of Theorem 1.

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2. Proof of Theorem 1. Our argument is based on the following four identities, each of which is valid for each complex number x such that $|x| < 1$:

$$\prod_1^\infty (1 + x^n)(1 - x^{2n-1}) = 1, \tag{3}$$

$$\prod_1^\infty (1 - x^n)(1 - x^{2n-1}) = \sum_{-\infty}^\infty (-x)^{n^2}, \tag{4}$$

$$\prod_1^\infty (1 - x^{2n})(1 + x^n) = \sum_0^\infty x^{n(n+1)/2}, \tag{5}$$

$$x\left\{\sum_0^\infty x^{n(n+1)/2}\right\}^8 = \sum_1^\infty \frac{n^3 x^n}{1 - x^{2n}}. \tag{6}$$

Identities (3)–(5), due to Euler and Gauss, are now classical; e.g., see [**1**, pp. 277–284]. Identity (6) is *not* as familiar as the others; see [**2**, p. 144].

With the help of (3) we express (5) as follows:

$$\prod_1^\infty (1 - x^n)(1 - x^{2n-1})^{-2} = \sum_0^\infty x^{n(n+1)/2}.$$

We then multiply the eighth power of this identity by the sixteenth power of identity (4) to get

$$\prod_1^\infty (1 - x^n)^{24} = \left\{\sum_0^\infty x^{n(n+1)/2}\right\}^8 \left\{\sum_{-\infty}^\infty (-x)^{n^2}\right\}^{16}.$$

Now, we multiply the foregoing identity by x, appeal to identity (6) and the definitions of τ and r_{16} to write

$$\sum_1^\infty \tau(n) x^n = \sum_1^\infty \frac{n^3 x^n}{1 - x^{2n}} \cdot \sum_0^\infty (-1)^n r_{16}(n) x^n. \tag{7}$$

But,

$$\begin{aligned}\sum_1^\infty \frac{n^3 x^n}{1 - x^{2n}} &= \sum_{n=1}^\infty n^3 x^n \sum_{k=0}^\infty x^{2nk} = \sum_{n=1}^\infty \sum_{k=0}^\infty n^3 x^{n(2k+1)} \\ &= \sum_{m=1}^\infty x^m \sum_{\substack{d|m \\ d \text{ odd}}} \left(\frac{m}{d}\right)^3 = \sum_{m=1}^\infty 2^{3b(m)} \sigma_3(O(m)) x^m.\end{aligned}$$

Hence, (7) becomes

$$\begin{aligned}\sum_1^\infty \tau(n) x^n &= \sum_{i=1}^\infty 2^{3b(i)} \sigma_3(O(i)) x^i \cdot \sum_{j=0}^\infty (-1)^j r_{16}(j) x^j \\ &= \sum_1^\infty x^n \sum_1^n (-1)^{n-i} r_{16}(n-i) 2^{3b(i)} \sigma_3(O(i)).\end{aligned}$$

Comparing coefficients of x^n, we thus prove our theorem.

3. Congruence properties of τ.

COROLLARY 2. *For each positive integer n,*

$$\tau(n) \equiv 2^{3b(n)}\sigma_3(O(n)) \pmod{32}.$$

PROOF. First of all, we observe that for each positive integer m, $r_{16}(m) \equiv 0 \pmod{32}$. We see this most easily by observing that

$$\sum_0^\infty r_{16}(m)x^m = \left[\left(\sum_{-\infty}^{\infty} x^{n^2}\right)^8\right]^2 = \sum_0^\infty x^m \sum_0^m r_8(m-k)r_8(k),$$

so that

$$r_{16}(m) = \sum_0^m r_8(m-k)r_8(k), \qquad m \geqslant 0.$$

But, for each positive integer n,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3,$$

e.g., see [**1**, p. 314]. Hence, it is now clear that for positive m, 32 divides $r_{16}(m)$. Thus, for each positive integer n, we rewrite (2) as

$$\begin{aligned}\tau(n) &= 2^{3b(n)}\sigma_3(O(n)) + \sum_{i=1}^{n-1} (-1)^{n-i} r_{16}(n-i)2^{3b(i)}\sigma_3(O(i)) \\ &\equiv 2^{3b(n)}\sigma_3(O(n)) \pmod{32}.\end{aligned}$$

COROLLARY 3. *For each positive integer n,*

$$\tau(n) \equiv 0 \pmod 8 \quad or \quad \tau(n) \equiv \sigma(n) \pmod 8$$

according as n is even or odd.

PROOF. Let n denote an arbitrary positive integer. Then, by Corollary 2

$$\tau(n) \equiv 2^{3b(n)}\sigma_3(O(n)) \pmod 8.$$

If (i) n is even, then $b(n) > 0$, whence $\tau(n) \equiv 0 \pmod 8$. If (ii) n is odd, then $b(n) = 0$, whence $\tau(n) \equiv \sigma_3(n) \pmod 8$. But, for each integer k,

$$(2k+1)^3 \equiv 2k+1 \pmod 8,$$

whence

$$\sigma_3(2k+1) \equiv \sigma(2k+1) \pmod 8.$$

Hence, if n is odd, then

$$\tau(n) \equiv \sigma(n) \pmod 8.$$

COROLLARY 4. *For each positive integer n, $\tau(n)$ is odd, if and only if n is an odd square.*

PROOF. This is proved using Corollary 3 and the well-known fact: $\sigma(n)$ is odd, if and only if n is a square or twice a square.

CONCLUDING REMARKS. It seems likely that the representation (2) of τ will render the asymptotical behavior of this important function along somewhat more accessible lines.

REFERENCES

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