A FORMULA FOR RAMANUJAN'S TAU FUNCTION

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Abstract. A formula for Ramanujan's tau function $\tau$, defined by $\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}$ ($|x| < 1$), is presented. The author then observes that some of the known congruence properties of $\tau$ are immediate consequences of this formula representation.

1. Introduction. The subject of this investigation is the arithmetical function $\tau$ defined by the identity

$$\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24},$$

valid for each complex number $x$ such that $|x| < 1$. As intimated in the above title, S. Ramanujan [2, p. 151] was the first mathematician to consider this function. We here propose to derive a formula which expresses $\tau$ in terms of two simpler arithmetical functions. Accordingly, we define these functions, as follows:

Definition. Let $k$ denote an arbitrary positive integer. Then, for each positive integer $n$: (i) $\sigma_k(n)$ denotes the sum of the $k$th powers of all positive divisors of $n$ (of course, $\sigma_1 = \sigma$, the ordinary sum-of-divisors function); (ii) $b(n)$ denotes the exponent of the highest power of 2 dividing $n$; and $O(n)$ is then defined by the equation $n = 2^{b(n)}O(n)$. (Hence, $b(n)$ is a nonnegative integer and $O(n)$ is odd.) Finally, (iii) for each nonnegative integer $n$, $r_k(n)$ denotes the cardinal number of the set

$$\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k | n = x_1^2 + x_2^2 + \cdots + x_k^2\}.$$  

(Note that $r_k(0) = 1$.)

We are now prepared to state our main result.

Theorem 1. For each positive integer $n$,

$$\tau(n) = \sum_{i=1}^{n} (-1)^{n-i} r_{16}(n - i) 2^{3b(i)} \sigma_3(O(i)).$$

The two simpler functions are then $\sigma_3$ and $r_{16}$. (The functions $b$ and $O$ are trivially defined by the fundamental theorem of arithmetic.) In §2 we prove this theorem. §3 is devoted to a discussion of some of the known congruence properties of $\tau$. As a matter of fact, we restrict attention to those properties which are immediate consequences of Theorem 1.

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2. Proof of Theorem 1. Our argument is based on the following four identities, each of which is valid for each complex number $x$ such that $|x| < 1$:

(3) \[ \prod_{l}^{\infty} (1 + x^n)(1 - x^{2n-1}) = 1, \]

(4) \[ \prod_{l}^{\infty} (1 - x^n)(1 - x^{2n-1}) = \sum_{-\infty}^{\infty} (-x)^n, \]

(5) \[ \prod_{l}^{\infty} (1 - x^{2n})(1 + x^n) = \sum_{0}^{\infty} x^{n(n+1)/2}, \]

(6) \[ x \left( \sum_{0}^{\infty} x^{n(n+1)/2} \right)^8 = \sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}}. \]

Identities (3)–(5), due to Euler and Gauss, are now classical; e.g., see [1, pp. 277–284]. Identity (6) is not as familiar as the others; see [2, p. 144].

With the help of (3) we express (5) as follows:

\[ \prod_{l}^{\infty} (1 - x^n)(1 - x^{2n-1})^{-2} = \sum_{0}^{\infty} x^{n(n+1)/2}. \]

We then multiply the eighth power of this identity by the sixteenth power of identity (4) to get

\[ \prod_{l}^{\infty} (1 - x^n)^{24} = \left( \sum_{0}^{\infty} x^{n(n+1)/2} \right)^8 \left( \sum_{-\infty}^{\infty} (-x)^n \right)^{16}. \]

Now, we multiply the foregoing identity by $x$, appeal to identity (6) and the definitions of $\tau$ and $r_{16}$ to write

(7) \[ \sum_{1}^{\infty} \tau(n)x^n = \sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}} \cdot \sum_{0}^{\infty} (-1)^n r_{16}(n) x^n. \]

But,

\[ \sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}} = \sum_{n=1}^{\infty} n^3 x^n \sum_{k=0}^{\infty} x^{2nk} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n^3 x^{n(2k+1)} \]

\[ = \sum_{m=1}^{\infty} x^m \sum_{d|m} \left( \frac{m}{d} \right)^3 = \sum_{m=1}^{\infty} 2^{3b(m)} \sigma_3(O(m)) x^m. \]

Hence, (7) becomes

\[ \sum_{1}^{\infty} \tau(n)x^n = \sum_{i=1}^{\infty} 2^{3b(i)} \sigma_3(O(i)) x^i \cdot \sum_{j=0}^{\infty} (-1)^j r_{16}(j) x^j \]

\[ = \sum_{1}^{n} x^n \sum_{i=1}^{n} (-1)^{n-i} r_{16}(n - i) 2^{3b(i)} \sigma_3(O(i)). \]

Comparing coefficients of $x^n$, we thus prove our theorem.
3. Congruence properties of \( \tau \).

**Corollary 2.** For each positive integer \( n \),
\[
\tau(n) \equiv 2^{3b(n)}\sigma_3(O(n)) \pmod{32}.
\]

**Proof.** First of all, we observe that for each positive integer \( m \), \( r_{16}(m) \equiv 0 \pmod{32} \). We see this most easily by observing that
\[
\sum_{k=0}^{\infty} r_{16}(m) x^k = \left[ \left( \sum_{n=0}^{\infty} x^{n^2} \right)^{8} \right]^2 = \sum_{k=0}^{\infty} x^m \sum_{k=0}^{m} r_{8}(m-k)r_{8}(k),
\]
so that
\[
r_{16}(m) = \sum_{k=0}^{m} r_{8}(m-k)r_{8}(k), \quad m \geq 0.
\]
But, for each positive integer \( n \),
\[
r_{8}(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3,
\]
e.g., see [1, p. 314]. Hence, it is now clear that for positive \( m \), 32 divides \( r_{16}(m) \). Thus, for each positive integer \( n \), we rewrite (2) as
\[
\tau(n) = 2^{3b(n)}\sigma_3(O(n)) + \sum_{i=1}^{n-1} (-1)^{n-i} r_{16}(n-i) 2^{3b(i)}\sigma_3(O(i))
\]
\[
\equiv 2^{3b(n)}\sigma_3(O(n)) \pmod{32}.
\]

**Corollary 3.** For each positive integer \( n \),
\[
\tau(n) \equiv 0 \pmod{8} \quad \text{or} \quad \tau(n) \equiv \sigma(n) \pmod{8}
\]
according as \( n \) is even or odd.

**Proof.** Let \( n \) denote an arbitrary positive integer. Then, by Corollary 2
\[
\tau(n) \equiv 2^{3b(n)}\sigma_3(O(n)) \pmod{8}.
\]
If (i) \( n \) is even, then \( b(n) > 0 \), whence \( \tau(n) \equiv 0 \pmod{8} \). If (ii) \( n \) is odd, then \( b(n) = 0 \), whence \( \tau(n) \equiv \sigma_3(n) \pmod{8} \). But, for each integer \( k \),
\[
(2k + 1)^3 \equiv 2k + 1 \pmod{8},
\]
whence
\[
\sigma_3(2k + 1) \equiv \sigma(2k + 1) \pmod{8}.
\]
Hence, if \( n \) is odd, then
\[
\tau(n) \equiv \sigma(n) \pmod{8}.
\]

**Corollary 4.** For each positive integer \( n \), \( \tau(n) \) is odd, if and only if \( n \) is an odd square.

**Proof.** This is proved using Corollary 3 and the well-known fact: \( \sigma(n) \) is odd, if and only if \( n \) is a square or twice a square.
Concluding Remarks. It seems likely that the representation (2) of \( \tau \) will render the asymptotical behavior of this important function along somewhat more accessible lines.

References


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