A FORMULA FOR RAMANUJAN'S TAU FUNCTION

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Abstract. A formula for Ramanujan's tau function τ, defined by ∑_n τ(n) x^n = x ∏_n (1 - x^n)^24 (|x| < 1), is presented. The author then observes that some of the known congruence properties of τ are immediate consequences of this formula representation.

1. Introduction. The subject of this investigation is the arithmetical function τ defined by the identity

\[ \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \]

valid for each complex number x such that |x| < 1. As intimated in the above title, S. Ramanujan [2, p. 151] was the first mathematician to consider this function. We here propose to derive a formula which expresses τ in terms of two simpler arithmetical functions. Accordingly, we define these functions, as follows:

Definition. Let k denote an arbitrary positive integer. Then, for each positive integer n: (i) \( \sigma_k(n) \) denotes the sum of the \( k \)th powers of all positive divisors of \( n \) (of course, \( \sigma_1 = \sigma \), the ordinary sum-of-divisors function); (ii) \( b(n) \) denotes the exponent of the highest power of 2 dividing \( n \); and \( O(n) \) is then defined by the equation \( n = 2^{b(n)} O(n) \). (Hence, \( b(n) \) is a nonnegative integer and \( O(n) \) is odd.) Finally, (iii) for each nonnegative integer \( n \), \( r_k(n) \) denotes the cardinal number of the set

\[ \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k | n = x_1^2 + x_2^2 + \cdots + x_k^2\}. \]

(Note that \( r_k(0) = 1 \).)

We are now prepared to state our main result.

Theorem 1. For each positive integer \( n \),

\[ \tau(n) = \sum_{i=1}^{n} (-1)^{n-i} r_{16}(n-i) 2^{3b(i)} \sigma_3(O(i)). \]

The two simpler functions are then \( \sigma_3 \) and \( r_{16} \). (The functions \( b \) and \( O \) are trivially defined by the fundamental theorem of arithmetic.) In §2 we prove this theorem. §3 is devoted to a discussion of some of the known congruence properties of \( \tau \). As a matter of fact, we restrict attention to those properties which are immediate consequences of Theorem 1.
2. Proof of Theorem 1. Our argument is based on the following four identities, each of which is valid for each complex number \( x \) such that \( |x| < 1 \):

\[
\prod_{l=1}^{\infty} (1 + x^n)(1 - x^{2n-1}) = 1,
\]

\[
\prod_{l=1}^{\infty} (1 - x^n)(1 - x^{2n-1}) = \sum_{-\infty}^{\infty} (-x)^n,
\]

\[
\prod_{l=1}^{\infty} (1 - x^{2n})(1 + x^n) = \sum_{0}^{\infty} x^{n(n+1)/2},
\]

\[
x \left( \sum_{0}^{\infty} x^{n(n+1)/2} \right)^8 = \sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}}.
\]

Identities (3)–(5), due to Euler and Gauss, are now classical; e.g., see [1, pp. 277–284]. Identity (6) is not as familiar as the others; see [2, p. 144].

With the help of (3) we express (5) as follows:

\[
\prod_{l=1}^{\infty} (1 - x^n)(1 - x^{2n-1})^{-2} = \sum_{0}^{\infty} x^{n(n+1)/2}.
\]

We then multiply the eighth power of this identity by the sixteenth power of identity (4) to get

\[
\prod_{l=1}^{\infty} (1 - x^n)^{24} = \left( \sum_{0}^{\infty} x^{n(n+1)/2} \right)^8 \left( \sum_{-\infty}^{\infty} (-x)^n \right)^{16}.
\]

Now, we multiply the foregoing identity by \( x \), appeal to identity (6) and the definitions of \( \tau \) and \( r_{16} \) to write

\[
\sum_{1}^{\infty} \tau(n) x^n = \sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}} \cdot \sum_{0}^{\infty} (-1)^n r_{16}(n) x^n.
\]

But,

\[
\sum_{1}^{\infty} \frac{n^3 x^n}{1 - x^{2n}} = \sum_{n=1}^{\infty} n^3 x^n \sum_{k=0}^{\infty} x^{2nk} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n^3 x^{2(k+1)} = \sum_{m=1}^{\infty} x^m \sum_{d|m} \left( \frac{m}{d} \right)^3 = \sum_{m=1}^{\infty} \sum_{d|\text{odd}} 2^{3b(m)} \sigma_3(O(m)) x^m.
\]

Hence, (7) becomes

\[
\sum_{1}^{\infty} \tau(n) x^n = \sum_{i=1}^{\infty} 2^{3b(i)} \sigma_3(O(i)) x^i \cdot \sum_{j=0}^{\infty} (-1)^j r_{16}(j) x^j = \sum_{1}^{\infty} x^n \sum_{1}^{\infty} (-1)^{n-i} r_{16}(n-i) 2^{3b(i)} \sigma_3(O(i)).
\]

Comparing coefficients of \( x^n \), we thus prove our theorem.
3. Congruence properties of \( \tau \).

**Corollary 2.** For each positive integer \( n \),
\[
\tau(n) \equiv 2^{3b(n)}a_3(O(n)) \pmod{32}.
\]

**Proof.** First of all, we observe that for each positive integer \( m \), \( r_{16}(m) \equiv 0 \pmod{32} \). We see this most easily by observing that
\[
\sum_{n=0}^{\infty} r_{16}(m)x^n = \left( \sum_{n=0}^{\infty} x^n \right)^8 = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} r_8(m-k) r_8(k),
\]
so that
\[
r_{16}(m) = \sum_{k=0}^{m} r_8(m-k) r_8(k), \quad m \geq 0.
\]

But, for each positive integer \( n \),
\[
r_8(n) = 16(-1)^n \sum_{d | n} (-1)^d d^3,
\]
e.g., see [1, p. 314]. Hence, it is now clear that for positive \( m \), 32 divides \( r_{16}(m) \).

Thus, for each positive integer \( n \), we rewrite (2) as
\[
\tau(n) = 2^{3b(n)}a_3(O(n)) + \sum_{i=1}^{n-1} (-1)^{n-i} r_{16}(n-i) 2^{3b(i)} a_3(O(i))
\]
\[
\equiv 2^{3b(n)}a_3(O(n)) \pmod{32}.
\]

**Corollary 3.** For each positive integer \( n \),
\[
\tau(n) \equiv 0 \pmod{8} \quad \text{or} \quad \tau(n) \equiv \sigma(n) \pmod{8}
\]
according as \( n \) is even or odd.

**Proof.** Let \( n \) denote an arbitrary positive integer. Then, by Corollary 2
\[
\tau(n) \equiv 2^{3b(n)}a_3(O(n)) \pmod{8}.
\]
If (i) \( n \) is even, then \( b(n) > 0 \), whence \( \tau(n) \equiv 0 \pmod{8} \). If (ii) \( n \) is odd, then \( b(n) = 0 \), whence \( \tau(n) \equiv \sigma_3(n) \pmod{8} \). But, for each integer \( k \),
\[
(2k + 1)^3 \equiv 2k + 1 \pmod{8},
\]
whence
\[
\sigma_3(2k + 1) \equiv \sigma(2k + 1) \pmod{8}.
\]

Hence, if \( n \) is odd, then
\[
\tau(n) \equiv \sigma(n) \pmod{8}.
\]

**Corollary 4.** For each positive integer \( n \), \( \tau(n) \) is odd, if and only if \( n \) is an odd square.

**Proof.** This is proved using Corollary 3 and the well-known fact: \( \sigma(n) \) is odd, if and only if \( n \) is a square or twice a square.
CONCLUDING REMARKS. It seems likely that the representation (2) of $\tau$ will render the asymptotical behavior of this important function along somewhat more accessible lines.

REFERENCES


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