HYPERBOLICITY OF A COMPLEX MANIFOLD 
AND OTHER EQUIVALENT PROPERTIES

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Abstract. Defining the notions of Schottky, Landau and Picard properties on a 
plane domain, the first author [3] proved that a domain in C having any of these 
properties is equivalent to the hyperbolicity of the domain.

In this paper the authors extend these notions to higher-dimensional case and 
obtain other various equivalent conditions for the hyperbolicity of a complex 
manifold.

1. Introduction. Let M be a connected complex manifold of dimension m with 
hermitian metric hM. Let ρM denote the distance function associated with hM. 
Following S. Kobayashi [6], we define M to be (Kobayashi) hyperbolic if the 
Kobayashi pseudometric kM is a metric. In [8], H. Royden has constructed the 
infinitesimal form of the Kobayashi pseudometric

\[
K_M(p, \xi) = \inf \{ |v| : \exists f \in \mathcal{H}(\Delta, M) \ni f(0) = p, f'(0)v = \xi \}
\]

where (p, ξ) ∈ T(M), the tangent bundle of M, and \( \mathcal{H}(\Delta, M) \) denotes the class of 
all holomorphic maps f of the open disc Δ into M, and he has shown that if M is 
hyperbolic then \( k_M \) is the integrated form of \( K_M \). The notion of hyperbolicity of M 
can also be defined in terms of \( K_M \): A complex manifold M is hyperbolic if for each 
p ∈ M there exists a coordinate neighborhood U of p and a positive constant c such 
that for (q, v) ∈ T(U) ≈ U × \( \mathbb{C}^m \)

\[
K_M(q, v) > c|v|
\]

where \( |v| \) denotes the euclidean distance. This definition of hyperbolicity coincides 
with that of Kobayashi (see [8]). In the case of a hermitian manifold, the above 
condition is equivalent to the condition that for each p ∈ M there is a neighborhood 
U of p and a positive constant c such that

\[
K_M(q, v) \geq c h_M(q, v)
\]

for all (q, v) ∈ T(U).

Definition 1. A complex manifold M is said to satisfy the Schottky property if for 
each p ∈ M, for each relatively compact open set W in a coordinate neighborhood 
of p and an r ∈ (0, 1), there exists a positive constant S = S(W, r) such that every
holomorphic map \( f \in \mathcal{H}(\Delta, M) \) with \( f(0) \in W \) satisfies
\[
\rho_M(p, f(z)) \leq S \quad \text{for} \quad |z| \leq r.
\]

\( M \) satisfies the Landau property if for each \( p \in M \) and each relatively compact open set \( W \) in a coordinate neighborhood of \( p \), there exists a positive constant \( R = R(W) \) such that
\[
\sup\{h_M(f(0), f'(0)) : f \in \mathcal{H}(\Delta, M) \text{ with } f(0) \in W \} \leq R.
\]

Then we can prove the following.

**Theorem 1.** The following statements are equivalent on any hermitian manifold \( M \).

(a) \( M \) is hyperbolic.
(b) \( M \) satisfies the Schottky property.
(c) \( M \) satisfies the Landau property.

**Definition 2.** A map \( f \in \mathcal{H}(\Delta, M) \) is said to be Bloch (see [4]) if
\[
\sup\{Q_f(z) : z \in \Delta\} < \infty,
\]
where
\[
Q_f(z) = (1 - |z|^2)h_M(f(z), f'(z)).
\]

We remark that the nonnegative function \( Q_f \) is invariant under the group \( \text{Aut}(\Delta) \) of holomorphic automorphisms of \( \Delta \) in the sense that for all \( \varphi \in \text{Aut}(\Delta) \)
\[
Q_{f \circ \varphi}(z) = Q_f(\varphi(z)), \quad z \in \Delta.
\]

It is an easy consequence of the definition of Kobayashi metric that if (5a) holds for all \( f \in \mathcal{H}(\Delta, M) \), then \( M \) is hyperbolic. The converse is, however, not true in general (see lemma in §3). If \( M \) is compact, we obtain the following characterization of the hyperbolicity of \( M \).

**Theorem 2.** Let \( M \) be a connected compact hermitian manifold. The following statements are equivalent.

(a) \( M \) is hyperbolic.
(b) \( \sup\{h_M(f(0), f'(0)) : f \in \mathcal{H}(\Delta, M)\} < \infty \).
(c) \( \sup\{Q_f(z) : z \in \Delta\} < \infty \) for all \( f \in \mathcal{H}(\Delta, M) \), i.e., every \( f \in \mathcal{H}(\Delta, M) \) is Bloch.
(d) \( M \) has the Picard property, i.e., there is no nonconstant holomorphic map \( f : \mathbb{C} \to M \).

Unlike in the one complex-dimensional case [3], the notion of the Picard property is a weaker notion than the Schottky property, the Landau property or the hyperbolicity for a noncompact manifold [6].

We note that the notion of the Picard property used in [3] and here, somewhat reluctantly, coincides with the term previously used by S. Kobayashi [6] and others.

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2. Proof of Theorem 1. It will be proved through the following implications:

(a) ⇔ (b), (a) ⇒ (c) ⇒ (a).

To prove (a) ⇒ (b), let $p \in M$ and $U$ be its local coordinate neighborhood. Let $W$ be a relatively compact neighborhood of $p$ in $U$. Since $k_M$ is a metric, it induces the standard topology in $M$ [1]. Therefore, we may define $W = \{q: k_M(p, q) < \rho_0\}$ for some $\rho_0 > 0$ and $U = \{q: k_M(p, q) < \rho_0 + \rho_1\}$, where $\rho_1 = \tan^{-1}r$, for any given $r \in (0, 1)$. If $f \in \mathcal{O}(\Delta, M)$ satisfies $f(0) \in W$, then $k_M(p, f(0)) < \rho_0$ and $k_M(f(0), f(z)) \leq k_M(0, z) \leq \rho_0(0, r) = \tan^{-1}r = \rho_1$ for $|z| \leq r$. Therefore, it follows from the triangle inequality that $k_M(p, f(z)) < \rho_0 + \rho_1$ whenever $|z| < r$. Since $k_M$ is a metric, $k_M \geq c\rho_M$ for some $c > 0$ which proves (b) with $S = (\rho_0 + \rho_1)/c$.

For the proof of $(b) \Rightarrow (a)$, we follow the method of [5]: Let $p$ and $q$ be two distinct points of $M$. We may take a coordinate neighborhood $U$ of $p$ which does not contain $q$. Let $\varphi: U \to B = \{w \in \mathbb{C}^m: ||w|| < 1\}$ be a biholomorphic map with $\varphi(p) = 0$, $\varphi(U) = B$. Without loss of generality, we may assume that $\varphi(W) = \{w \in \mathbb{C}^m: ||w|| < \rho) = B_\rho$ for some $\rho \in (0, 1)$ and $\varphi(V) \subset B$, $V = \{m \in M: \rho_M(p, M) < S\}$. Let $r \in (0, 1)$ be given. If $|z| < r/2$, then there exists a constant $c > 0$ such that $\rho_M(0, z) \geq c\rho_M(0, z)$ or, equivalently, $\rho_M(0, z) \geq c\rho_M(0, z)$. Let $\alpha = (p = p_0, p_1, \ldots, p_l = q; a_1, \ldots, a_l; f_1, \ldots, f_l)$ be any chain connecting $p$ and $q$, used in the construction of the Kobayashi metric $k_M(p, q)$ [6]. Since $\varphi(q)$ lies outside $B$, there must be an index, say $l_0$, $0 < l_0 < l$, such that $\varphi(p_0), \varphi(p_1), \ldots, \varphi(p_{l-1}) \in B$ and $\varphi(p_{l_0}) \notin B$. By the homogeneity of $\Delta$, we may assume that $a_1, \ldots, a_{l_0}$ lie in $\Delta_{r/2}$. Then

\[
|\alpha| \geq \sum_{i=1}^{l_0} \rho_M(0, a_i) \geq c \sum_{i=1}^{l_0} \rho_M(0, a_i) \geq c \sum_{i=1}^{l_0} k_B(\varphi(p_{i-1}), \varphi(p_i))
\]

\[
\geq ck_B(0, \varphi(p_{l_0})) = c \tan^{-1}||\varphi(p_{l_0})|| \geq c \tan^{-1}r = c'.
\]

Thus, $k_M(p, q) \geq c' > 0$. Here we used the fact that on $B$ the Kobayashi metric agrees with the standard Kähler metric. It is given by

\[
k_B(0, z) = \frac{1}{2} \log \frac{1 + ||z||}{1 - ||z||} = \tan^{-1}||z|| \quad \text{for } z \in B.
\]

See [4].

(a) ⇒ (c). Suppose that $M$ fails to satisfy the Landau property. Then there exist a point $p_0$ in an open neighborhood $W$ which is relatively compact in a local coordinate neighborhood $U$ of $p_0$ and a sequence $f_k \in \mathcal{O}(\Delta, M)$ such that $f_k(0) \in W$ and $h_M(f_k(0), f_k(0)) \to \infty$ as $k \to \infty$. We claim that if $M$ is hyperbolic then there exists an $r \in (0, 1)$ such that $\{f_k\}$ contains a subsequence which converges uniformly to a holomorphic map $f: \Delta_r \to M$. First we observe that $\overline{W} \cap (M \setminus U) = \varnothing$ where $\overline{W}$ is compact and $M \setminus U$ is closed. Therefore, there exists a number $\rho' > 0$ such that

\[
Q = \{p \in M: k_M(\overline{W}, p) < \rho'\} \subset \subset U.
\]

Since $k_M(f_k(0), f_k(z)) \leq \rho_M(0, z) = \tan^{-1}|z|$ for all $f_k \in \mathcal{O}(\Delta, M)$ with $f_k(0) \in W$, $f_k(z) \in Q$ whenever $|z| \leq r' = \tan h\rho'$. Since $Q$ is relatively compact, it is bounded in $\mathbb{C}^m$. Therefore, by Montel’s theorem there is a subsequence of $\{f_k\}$ which
converges uniformly on $\Delta_r$ to a holomorphic map $f: \Delta_r \to M$ for $r < r'$. Denote again this convergent subsequence by $(f_k)$. By Weierstrass' theorem, $f'_k(z)$ converges to $f'(z)$ uniformly on $\Delta_r$. In particular, $f'_k(0)$ converges to $f'(0)$. Thus, $h_M(f_k(0), f'_k(0)) \to h_M(f(0), f'(0)) < \infty$ as $k \to \infty$, which is a contradiction.

(c) $\Rightarrow$ (a). Given $(q, \xi) \in T(W)$ and $v \in C$, let $f \in \mathcal{K}(\Delta, M)$ satisfy $f(0) = q$ and $f'(0)v = \xi$. Then by (4) of §1
\[ R|v| \geq h_M(f(0), f'(0))|v| = h_M(f(0), f'(0)v) = h_M(q, \xi) \]
or
\[ |v| \geq \frac{1}{R} h_M(q, \xi), \]
where $R$ is the upper bound given in (4) of §1. Thus,
\[ e_{f(0), \xi} > ch_M(q, \xi), \]
eq 1/R, as desired.

3. Proof of Theorem 2. By a result of R. Brody [2], (a) $\Leftrightarrow$ (d) holds on a connected compact manifold. Therefore, it is enough to prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). First we prove the following

**Lemma.** A complex manifold $M$ is hyperbolic if $\sup\{Q_f(z): z \in \Delta\} < \infty$ for all $f \in \mathcal{K}(\Delta, M)$. The converse is not true in general.

**Proof.** The above hypothesis implies that there exists a constant $Q > 0$ such that for all $f \in \mathcal{K}(\Delta, M)$
\[ h_M(f(z), f'(z)v) \leq QK_\Delta(z, v) \quad (z \in \Delta, v \in C). \]

Let $\tilde{h}_M$ be the distance function associated with the differential metric $\tilde{h}_M = Q^{-1}h_M$. By integrating both sides of (1) along the geodesic curve between two points $z$ and $w$ in $\Delta$ and using the definition of integrated distance $\tilde{h}_M$, we have
\[ \tilde{h}_M(f(z), f(w)) \leq k_\Delta(z, w) \]
for all $f \in \mathcal{K}(\Delta, M)$. Since $k_M$ is the largest among those pseudometrics which satisfy (2), $\tilde{h}_M \leq k_M$ (see [6]). Thus, $M$ is hyperbolic.

To show the second half, let $M$ be the right half-plane of $C$ and $h_M$ the euclidean metric. Then $M$ is clearly hyperbolic. Set
\[ f(z) = (1 + z)/(1 - z) \quad \text{and} \quad z_n = 1 - 1/n \in \Delta. \]
Then $f \in \mathcal{K}(\Delta, M)$ and
\[ Q_f(z_n) = (1 - |z_n|^2)|f'(z_n)| = 2(2n - 1) \to \infty \]
as $n \to \infty$. This completes the lemma.

From this lemma, (c) $\Rightarrow$ (a) trivially follows. To prove (a) $\Rightarrow$ (b), let $M$ be hyperbolic. By (c) of Theorem 1, for each $p \in M$ there exists a relatively compact open neighborhood $W_p$ of $p$ such that
\[ \sup\{||f'(0)||: f \in \mathcal{K}(\Delta, M), f(0) \in W_p\} < \infty. \]
Since \( M \) is compact, it can be covered by a finite number of such \( W_p \)'s. Thus, (b) follows.

(b) \( \Rightarrow \) (c). Suppose (c) fails to hold. Then there exist sequences \( (z_n) \) in \( \Delta \) and \( (f_n) \) in \( \mathcal{H}(\Delta, M) \) such that \( Q_{f_n}(z_n) > n \) for all \( n \). Since \( \Delta \) is homogeneous, for each \( z_n \) there exists \( \varphi_n \in \text{Aut}(\Delta) \) such that \( \varphi_n(0) = z_n \). Therefore, by the invariant property of \( Q_f \) (see (6) of §1),

\[
Q_{f_n}(z_n) = Q_{f_n}(\varphi_n(0)) = Q_{\varphi_n}(0) = h_M(g_n(0), g_n'(0)),
\]

where \( g_n = f_n \circ \varphi_n \). Since \( h_M \) is hermitian, it follows from the compactness of \( M \), condition (b) and (5) that \( Q_{f_n}(z_n) \) is bounded for all \( n \), which is a contradiction.

REFERENCES


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