EXTREME POINTS OF SUBORDINATION FAMILIES WITH UNIVALENT MAJORANTS

DAVID J. HALLENBECK

Abstract. Let \( s(F) \) denote the set of functions subordinate to a univalent function \( F \) in \( A \) the unit disc. Let \( B_0 \) denote the set of functions \( \phi(z) \) analytic in \( A \) satisfying \( |\phi(z)| < 1 \) and \( \phi(0) = 0 \). We prove the following results: If \( f = F \circ \phi \) is an extreme point of \( s(F) \) and \( F(\Delta) \) is a Jordan domain, then \( \phi \) is an extreme point of \( B_0 \).

1. Introduction. Let \( \Delta = \{ z : |z| < 1 \} \) and let \( \mathcal{S} \) denote the set of functions analytic in \( \Delta \). Let \( B_0 \) consist of the subset of \( \mathcal{S} \) consisting of all functions \( \phi \) that satisfy the conditions \( |\phi(z)| < 1 \) and \( \phi(0) = 0 \). Let \( EB_0 \) denote the extreme points of \( B_0 \). It is a classical fact [2, p. 125] that \( \phi \in EB_0 \) if and only if \( \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty \). Let \( S \) denote the subset of \( \mathcal{S} \) consisting of univalent functions \( f \) so that \( f(z) = z + \cdots \) in \( \Delta \).

Let \( F \) be in \( \mathcal{S} \) and be univalent in \( \Delta \). Let \( s(F) \) denote the subset of \( \mathcal{S} \) consisting of functions \( f \) that are subordinate to \( F \) in \( \Delta \). This means that \( f \in \mathcal{S} \), \( f(0) = F(0) \), and \( f(\Delta) \subset F(\Delta) \). These conditions are equivalent to the existence of \( \phi \in B_0 \) so that \( f = F \circ \phi \). Note that \( s(F) = \{ F \circ \phi : \phi \in B_0 \} \).

Let \( D \) denote \( F(\Delta) \). Since \( F \in H^p \) for \( p < \frac{1}{2} \) [2, p. 50], if we define \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \), then \( f(e^{i\theta}) \) exists almost everywhere. Let \( \lambda(\theta) \) denote the distance between \( f(e^{i\theta}) \) and \( \partial D \) where \( \partial D \) denotes the boundary of \( D \).

We let \( HS(F) \) and \( EHs(F) \) denote respectively the closed convex hull of \( s(F) \) and the extreme points of the closed convex hull of \( s(F) \). Since \( s(F) \) is compact, \( EHs(F) \subset s(F) \). Let \( Es(F) \) denote the extreme points of \( s(F) \). In [1], Yusuf Abu-Muhanna proved if \( F' \) is Nevanlinna and \( D \) a Jordan domain subset to a half plane then \( EHs(F) \subset \{ F \circ \phi : \phi \in EB_0 \} \). He conjectured that the inclusion holds for any univalent function \( F \). We prove this inclusion only assuming \( F \) is univalent and \( D \) is a Jordan domain. Our arguments are similar to those of Abu-Muhanna [1].

2. Functions subordinate to a univalent function. We let \( d(z, \Gamma) \) denote the distance between \( z \) and a closed set \( \Gamma \), \( m(A) \) denote the Lebesgue measure of \( A \) and \( \log^+ x = \max(0, \log x) \).

Theorem 1. Let \( F \) be in \( \mathcal{S} \) and be univalent in \( \Delta \). Whenever \( g \) is in \( s(F) \), let \( \lambda(\theta) \) denote the distance between \( g(e^{i\theta}) \) and \( \partial D \) where \( D = F(\Delta) \). Then when \( g = F \circ \phi \),

(a) \( \int_0^{2\pi} \log^+ \lambda(\theta) \, d\theta \) is convergent and

Received by the editors June 9, 1983.
1980 Mathematics Subject Classification. Primary 30C80; Secondary 30C55.
Key words and phrases. Analytic functions, bounded function, extreme point, Jordan domains, Nevanlinna class, subordinations, univalent function.
(b) \( \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \) implies \( \int_{0}^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty \).

If \( F' \) is Nevanlinna then

(c) \( \int_{0}^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty \) implies \( \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \).

**Proof.** Since \( F \) is univalent, it follows from [3, p. 22] that

\[
\frac{1}{4} (1 - |z|^2)|F'(z)| \leq d(F(z), \partial D) \leq (1 - |z|^2)|F'(z)|
\]

for all \( z \) in \( \Delta \). For \( g \) in \( s(F) \) we have \( g = F \circ \phi \) where \( \phi \in B_0 \). If \( |\phi(e^{i\theta})| = 1 \) for some \( \theta \) the univalence of \( F \) gives \( \lambda(\theta) = 0 \) and \( \log^+ \lambda(\theta) = 0 \). When \( g(e^{i\theta}) \) and \( \phi(e^{i\theta}) \) exist, and \( |\phi(e^{i\theta})| < 1 \), we obtain from (1)

\[
\frac{1}{4} (1 - |\phi(e^{i\theta})|^2)|F'(\phi(e^{i\theta}))| \leq \lambda(\theta) \leq (1 - |\phi(e^{i\theta})|^2)|F'(\phi(e^{i\theta}))|,
\]

which implies \( \lambda(\theta) \leq 2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \). We may assume, without loss of generality, that \( F \in S \) and so, it follows that

\[
|F'(z)| \leq \left| \frac{F(z)}{z} \right| \frac{1 + |z|}{1 - |z|}
\]

for \( z \) in \( [3, p. 21] \). Hence,

\[
2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \leq 4 \left| \frac{F(\phi(e^{i\theta}))}{\phi(e^{i\theta})} \right|
\]

and so

\[
0 \leq \log^+ \lambda(\theta) \leq \log^+ 4 \left| \frac{F(\phi(e^{i\theta}))}{\phi(e^{i\theta})} \right|.
\]

It is known that \( F(z)/z \) is in \( H^p \) for each \( p < \frac{1}{2} \) and so we have \( F(\phi(z))/\phi(z) \) in \( H^p \) for each \( p < \frac{1}{2} \). It follows that \( \log^+ |F(\phi(e^{i\theta}))/\phi(e^{i\theta})| \) is integrable on \( [0, 2\pi] \). It follows directly that (a) holds.

Now let \( A = \{ \theta : g(e^{i\theta}) \text{ exists and } \lambda(\theta) = 0 \} \). If \( m(A) > 0 \) then it follows that

\[
\int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty.
\]

It also follows that \( \int_{0}^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty \) since \( g(e^{i\theta}) \) on \( \partial D \) implies that \( |\phi(e^{i\theta})| = 1 \), given that \( F \in S \). Assume for the rest of the proof that \( m(A) = 0 \). This implies (2) holds for almost all \( \theta \). The fact that \( 1 \leq 1 + |\phi(e^{i\theta})| \leq 2 \) applied to (2) gives

\[
\frac{1}{4} (1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \leq \lambda(\theta) \leq 2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))|
\]

for almost all \( \theta \). Since \( F \) is in \( S \), we have [3, p. 21] \( |F'(z)| \geq (1 - |z|)/(1 + |z|)^3 \) for \( z \) in \( \Delta \). Hence

\[
|F'(\phi(e^{i\theta}))| \geq \frac{1}{8} (1 - |\phi(e^{i\theta})|)
\]

and this fact, taken together with the left-hand side of (3) gives

\[
\lambda(\theta) \geq \frac{1}{4} (1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \geq \frac{1}{32} (1 - |\phi(e^{i\theta})|)^2.
\]

This implies

\[
-\log 32 + 2 \log(1 - |\phi(e^{i\theta})|) \leq \log \lambda(\theta).
\]
Since \( \log^+ \lambda(\theta) \in L^1[0, 2\pi] \) we have
\[
-\infty \leq \int_0^{2\pi} \log \lambda(\theta) \, d\theta < M
\]
for some constant \( M \). If \( \int_0^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \) then (4) and (5) imply
\[
\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty
\]
and (b) holds.

The proof of (c) is given in [1, p. 440].

3. Jordan domains. The following lemma appeared in [1, p. 444].

**Lemma A.** Let \( D \) be a bounded Jordan domain. Let \( g \) be a nonconstant bounded analytic function in \( \Delta \). If \( g(e^{i\theta}) \in \overline{D} \) for almost all \( \theta \) then \( g(\theta) \subset D \).

**Lemma 1.** Let \( D \) be an unbounded Jordan domain. Let \( g \) be a nonconstant analytic function in \( \Delta \) so that \( g(\Delta) \cup D \) avoids an open set. If \( g(e^{i\theta}) \in \overline{D} \) for almost all \( \theta \) then \( g(\Delta) \subset D \).

**Proof.** There exists a complex number \( a \) and a \( \delta > 0 \) so that \( \{w: |w - a| < \delta\} \cap (g(\Delta) \cup D) = \emptyset \). It follows that the function \( h(z) = 1/(g(z) - a) \) is analytic and bounded for \( z \in \Delta \). Also, \( h(e^{i\theta}) = 1/(g(e^{i\theta}) - a) \) exists for almost all \( \theta \) and \( h(e^{i\theta}) \in \overline{f(D)} \) where \( f(w) = 1/(w - a) \). It follows from Lemma A that \( h(\Delta) \subset f(D) \) or \( f(g(\Delta)) \subset f(D) \). Hence \( g(\Delta) \subset D \).

**Theorem 2.** Let \( F \) be in \( \mathcal{E} \) and univalent in \( \Delta \) and suppose that \( D = F(\Delta) \) is a Jordan domain. Let \( \lambda(\theta) \) denote the distance between \( f(e^{i\theta}) \) and \( \partial D \) where \( f \in s(F) \). If \( f \in \mathcal{E}(F) \), then
\[
\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta = -\infty.
\]

Also, if \( F \) is bounded then (6) is equivalent to
\[
\int_0^{2\pi} \log \lambda(\theta) \, d\theta = -\infty.
\]

**Proof.** We assume
\[
\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta > -\infty
\]
and show that \( f \notin \mathcal{E}(F) \). Since \( \lambda(\theta)/(1 + \lambda(\theta)) < 1 \) our assumption implies by \( \log(\lambda(\theta)/(1 + \lambda(\theta))) \) is integrable on \([0, 2\pi]\). Let \( d(z; D) \) denote the distance between \( z \) and \( D \). Since \( D \) is a Jordan domain, there exists \( \epsilon, 0 < \epsilon < 1 \), such that \( E = \{z: d(z, D) \leq \epsilon\} \) avoids an open set. The function \( \log \frac{\epsilon}{2}(\lambda(\theta)/(1 + \lambda(\theta))) \) is also integrable on \([0, 2\pi]\). Let
\[
g(z) = z \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{\epsilon}{2} \frac{\lambda(t)}{1 + \lambda(t)} \, dt \right].
\]
(8) gives $|g(z)| < \varepsilon/2$ for $z \in \Delta$ and it is known that

$$|g(e^{i\theta})| = \frac{\varepsilon}{2} \frac{\lambda(\theta)}{1 + \lambda(\theta)}$$

for almost all $\theta$. Since $\lambda(\theta)$ is the distance between $f(e^{i\theta})$ and $\partial D$ and $|g(e^{i\theta})| < \lambda(\theta)/2$, it follows that $f(e^{i\theta}) \pm g(e^{i\theta}) \in \overline{D}$ for almost all $\theta$. Note that $g \neq 0$. If $h = f \pm g$ then $|g(e^{i\theta})| < \varepsilon/2$ implies $h(\Delta) \subset E$. Lemma 1 now implies $f(z) \pm g(z) \in D$ for all $z \in \Delta$. Since $g(0) = 0$ and $f \in s(F)$, we conclude $f \pm g \in s(F)$ and so $f \notin Es(F)$. This completes the proof.

**Remark.** Since $EHs(F) \subset Es(F)$, (6) holds if $f \in EHs(F)$.

**Theorem 3.** Let $F \in \mathcal{Q}$ be univalent in $\Delta$. If $D = F(\Delta)$ is a Jordan domain then $Es(F) \subset \{F \circ w: w \in EB_0\}$.

**Proof.** Let $g = F \circ w \in Es(F)$ where $w \in B_0$. We shall prove that

$$\int_{0}^{2\pi} \log(1 - |w(e^{i\theta})|) \, d\theta = -\infty$$

and so conclude that $w \in EB_0$. If we let $\lambda(\theta)$ be the distance between $g(e^{i\theta})$ and $\partial D$, then Theorem 2 gives

$$\int_{0}^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta = -\infty.$$

Let $E = \{\theta: \lambda(\theta) \text{ exist and } \lambda(\theta) \leq 1\}$ and let $G = \{\theta: \lambda(\theta) \text{ exist and } \lambda(\theta) > 1\}$. Clearly $m(E \cup G) = 2\pi$. For $\theta \in E$, we have

$$\frac{\lambda(\theta)}{2} \leq \frac{\lambda(\theta)}{1 + \lambda(\theta)} \leq \lambda(\theta),$$

and for $\theta \in G$, we have $1 + \lambda(\theta) < 2\lambda(\theta)$ and so

$$\frac{1}{2} < \frac{\lambda(\theta)}{1 + \lambda(\theta)} < 1.$$

Equation (10) implies

$$\int_{G} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta$$

is convergent. Therefore,

$$\int_{0}^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta = \int_{E} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} \, d\theta = -\infty$$

and by (9) $\int_{E} \log \lambda(\theta) \, d\theta = -\infty$. Now

$$\int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = \int_{G} \log \lambda(\theta) \, d\theta + \int_{E} \log \lambda(\theta) \, d\theta.$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
By Theorem 1, \( \int_{0}^{2\pi} \log^+ \lambda(\theta) \, d\theta \) is convergent and so \( \int_{c} \log \lambda(\theta) \, d\theta = \int_{c} \log^+ \lambda(\theta) \, d\theta \) is convergent. Since \( \int_{c} \log \lambda(\theta) \, d\theta = -\infty \) we conclude from (11) that \( \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \). Again, by Theorem 1, we conclude that
\[
\int_{0}^{2\pi} \log(1 - |w(e^{i\theta})|) \, d\theta = -\infty.
\]
Hence \( w \in EB_0 \).

**Remark.** We note that \( EHS(F) \subseteq \{ F \circ w : w \in EB_0 \} \) since \( EHS(F) \subseteq Es(F) \).

**References**


**Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19711**