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THE SPECTRAL DIAMETER IN BANACH ALGEBRAS

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ABSTRACT. The element a is in the center of the Banach algebra A modulo its radical if and only if there is an upper bound for the diameters of the spectra of $a - uau^{-1}$ for u invertible. Applications of this result are given to general Banach algebras and to the essential spectrum of operators on a Hilbert Space.

In this paper we relate the diameters of the spectra of elements of a Banach algebra A to commutativity properties of A and to the Jacobson radical, $\operatorname{rad}(A)$. For a in A we let $\delta(a)$ be the diameter of the spectrum of A, and $\rho(a)$ be the spectral radius of A. Also, we let $Z(A)$ be the center of A modulo its radical; that is, a belongs to $Z(A)$ if and only if $ar - ra \in \operatorname{rad}(A)$ for all r in A. For the algebras that usually occur in applications, like C^*-algebras, group algebras, the Calkin algebra, etc., the radical is $\{0\}$, so that $Z(A)$ is actually the center of A.

Our results about the spectral diameter are similar to results of Aupetit [**1**] and Zemánek [**10**] about the spectral radius, and in fact, are generalizations of these results (see Corollary (1.3) below). Our main result, Theorem (1.1), gives a technical characterization of $Z(A)$. In §2, we obtain a variety of consequences of this technical characterization. For instance, we show in Corollary (2.2), that if the Hilbert space operator T is not the sum of a compact operator and a scalar, then there is a quasinilpotent operator Q for which $T + Q$ has more than one point in its essential spectrum.

1. The basic result. Essentially all the results in this paper are easy consequences of the following Theorem.

THEOREM (1.1). *Suppose that A is a Banach algebra with identity and that a belongs to A. Then a belongs to $Z(A)$, the center of A modulo its Jacobson radical, if and only if* $\sup \delta(a - uau^{-1}) < \infty$, *where the* sup *is taken over the invertible elements u of A.*

Notice that if a belongs to $Z(A)$, then $a - uau^{-1} \in \operatorname{rad}(A)$ for all invertible u, so that $\delta(a - uau^{-1})$ is always 0. In fact, for all our results characterizing commutativity modulo the radical, the only nontrivial part of the characterization is to show that the given property implies commutativity modulo the radical.

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In the proof of Theorem (1.1), we will need the following fact about 2×2 complex matrices. Since I have not been able to locate a proof in the literature, I include a sketch of the proof.

LEMMA (1.2). *If the* 2×2 *complex matrix* A *is not a scalar multiple of the identity, then for all* $M > 0$ *there is a matrix* B *similar to* A *for which* $\delta(A - B) > M$.

PROOF OF LEMMA. The 2×2 nonscalar matrices A and B are similar if and only if they have the same trace and the same determinant. When they are similar, the trace of $A - B$ is 0, so that $\delta(A - B)$ is large precisely when $|\det(A - B)|$ is large.

Without loss of generality, we can assume that A is in Jordan canonical form. Suppose first that A has two distinct eigenvalues so that $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a \neq b$. Then for each t, the matrix $A(t) = \left(\begin{smallmatrix} a-t & 1 \\ d & b+t \end{smallmatrix}\right)$ is similar to A provided that $d = d(t)$ is chosen so that A and $A(t)$ have the same determinant. With this choice of d, $|\det(A - A(t))| = |t(b - a)|$, which can be made arbitrarily large.

Now suppose that A has a single eigenvalue, so that $A = \left(\begin{smallmatrix} a & 0 \\ 1 & a \end{smallmatrix}\right)$. The proof proceeds as above but with $A(t) = \left(\begin{smallmatrix} a-1 & -t \\ 1/t & a+1 \end{smallmatrix}\right)$. This completes the proof of the lemma.

PROOF OF THEOREM (1.1). Suppose that a does not belong to $Z(A)$. Since the radical is the intersection of the annihilators of the algebraically irreducible A-modules [**3**, Definition 13, p. 124], there is an irreducible module X for which the action of a does not commute with the action of all r in A. Hence [**8**, Lemma 6.5, p. 35] a does not act as a scalar multiplication on X; so there exists an e in X for which e and ae are linearly independent. We break the rest of the proof into two cases.

Case 1. e, ae and a^2e are linearly independent. Then, by Sinclair's extension [**8**, Theorem 6.7, p. 36] of the Jacobson-Rickart-Yood density theorem, for each scalar λ there is an invertible element $u = u(\lambda)$ in A for which $ue = e$, $uae = ae$, $ua^2e = a^2e - \lambda ae$. A straightforward calculation shows that $(a - uau^{-1})e = 0$ and $(a - uau^{-1})ae = \lambda ae$, so that 0 and λ are both in the spectrum of $a - uau^{-1}$. Since λ is arbitrary, this shows that $\sup \delta(a - uau^{-1})$ is infinite.

Case 2. The linear space X spanned e and ae is invariant under a. In this case we can apply Lemma (1.2) to find a linear transformation u on the space X for which $a - uau^{-1}$ has arbitrarily large spectral diameter. By Sinclair's extension of the density theorem we can assume that u is an invertible element of A. Thus again we have that $\sup \delta(a - uau^{-1})$ is infinite, so the theorem is proved.

We now show that the analogue of Theorem (1.1) for the spectral radius, which is the basic tool in Aupetit's [**1**, p. 52] and Zemánek's [**10**, Theorem 1.2, p. 6] studies of the spectral radius, is an easy consequence of Theorem (1.1).

COROLLARY (1.3). *If* a *belongs to the Banach algebra with identity* A, *then* a *belongs to* $Z(A)$ *if and only if* $\sup \rho(a - uau^{-1})$ *is finite.*

PROOF. We just need to observe that, for all a and u, we have $\delta(a - uau^{-1}) \leqslant 2\rho(a - uau^{-1})$.

We remark that in the proof of Sinclair's density theorem [**8**, p. 37], the invertible element u is an exponential of an element of A. Hence in Theorem (1.1) and in Corollary (1.3) the sups can be taken over exponentials, rather than over all invertibles.

It is sometimes useful to consider the commutators $au - ua$ as well as $a - uau^{-1}$ [**6**, p. 287; **1**, pp. 45–46]. We do this in the next theorem.

THEOREM (1.4). *Suppose that a belongs to the Banach algebra A. Then a belongs to $Z(A)$ if and only if* $\sup \delta(au - ua) < \infty$, *where the* sup *is taken over all u in A.*

PROOF. Since we can adjoin an identity to A without changing the values of $au - ua$, we will assume that A has an identity. We consider the same cases as in the proof of Theorem (1.1). In the first case, we observe that $(au - ua)e = 0$ and $(au - ua)ae = \lambda ae$, so, as before, $\sup \delta(au - ua) = \infty$. For the second case we need only prove the appropriate analogue of Lemma (1.2). Let $B = UAU^{-1}$, where U is an invertible 2×2 matrix. Since $AU - UA$ has zero trace, $\delta(AU - UA)$ is large precisely when $|\det(AU - UA)|$ is large. But

$$|\det(AU - UA)| = |\det(A - UAU^{-1})||\det U|$$

and

$$|\det(U^{-1}A - AU^{-1})| = |\det(U^{-1})||\det(A - UAU^{-1})|.$$

Hence either $|\det(AU - UA)|$ or $|\det(U^{-1}A - AU^{-1})|$ is larger than

$$|\det(A - UAU^{-1})|,$$

which can be arbitrarily large by Lemma (1.2). This completes the proof.

2. Applications. In this section we give a number of applications of our main result, Theorem (1.1). Many of these applications are similar to the applications of Corollary (1.3) given by Aupetit [**1**] and Zemánek [**10**], so that we will sometimes refer to their work for proofs. We first characterize those elements of a Banach algebra which are sums of a scalar plus an element of the radical and we then apply the characterization to Hilbert space operators.

THEOREM (2.1). *Suppose that a belongs to the Banach algebra with identity A. Then a is of the form $\lambda + r$ with $r \in \operatorname{rad}(A)$ if and only if $a + q$ has one-point spectrum for every quasinilpotent q.*

PROOF. It is easy to see that the spectrum is unchanged by perturbations by elements of the radical [**5**, Theorem (5.5.7), p. 97], and it follows easily from the same properties of irreducible modules used in the beginning of the proof of Theorem (1.1) that $a = \lambda + r$ for some $r \in \operatorname{rad}(A)$ if and only if $a \in Z(A)$ and $\sigma(a) = \{\lambda\}$ (cf. [**10**, p. 6]). Since 0 is quasinilpotent, we need only prove that if $\sigma(a) = \{\lambda\}$ and $a + q$ has one-point spectrum for all quasinilpotent q, then a belongs to $Z(A)$. This follows easily from Theorem (1.1), since $a - \lambda$ is quasinilpotent, so that for all invertible u we have $\delta(a - uau^{-1}) = \delta(a - u(a - \lambda)u^{-1}) = 0$.

COROLLARY (2.2). *Suppose that the Hilbert space operator T is not the sum of a scalar and a compact operator; then there is a quasinilpotent operator Q for which $T + Q$ has more than one point in its essential spectrum.*

PROOF. Since the Calkin algebra is semisimple, and the quasinilpotents in this algebra are the cosets of Riesz operators, it follows from Theorem (2.1) that there is

a Riesz operator R for which $T + R$ has more than one element in its essential spectrum. Using the West decomposition [**9**; **5**, Theorem (3.5.2), p. 51], we can write R as a sum $C + Q$ with C compact and Q quasinilpotent. Since $T + R$ and $T + Q$ have the same essential spectrum, this completes the proof.

The spectral radius analogue of Theorem (2.1) [**10**, Theorem 3.3, p. 16] implies the following weaker form of Corollary (2.2): if T is not compact, then there is a quasinilpotent Q for which $T + Q$ is not a Riesz operator. Actually in this case there is, for separable Hilbert space, a direct construction [**4**] of a nilpotent operator N with $T + N$ invertible. I do not know of any such direct construction which proves Corollary (2.2).

We now apply Theorem (1.1) to characterizations of commutativity (modulo the radical). We start with a "local subadditivity" characterization of $Z(A)$.

LEMMA (2.3). *The element a of the Banach algebra A belongs to $Z(A)$ if and only if there are nonnegative constants K and M for which $\delta(a + b) \leqslant K + M\delta(b)$ for all b in A.*

PROOF. Since all the conditions are unchanged when an identity is adjoined, we can suppose A has an identity. For a in $Z(A)$, $\delta(a + b) \leqslant \delta(a) + \delta(b)$. Suppose, conversely, that a satisfies the given inequality $\delta(a + b) \leqslant K + M\delta(b)$. Then $\delta(a - uau^{-1}) \leqslant K + M\delta(-uau^{-1}) = K + M\delta(a)$, so the lemma follows from Theorem (1.1).

We use the above lemma to obtain short proofs of some characterizations of the commutativity of $A/\mathrm{rad}(A)$ given by Aupetit [**1**, Theorem 2, pp. 49–50].

THEOREM (2.4). *For a Banach algebra A the following are equivalent:*

(A) *$A/\mathrm{rad}(A)$ is commutative.*

(B) *The spectral diameter is subadditive.*

(C) *The spectral diameter is uniformly continuous.*

PROOF. Again we can, without loss of generality, assume that A has an identity. It is clear that (A) implies (B) and (C). Lemma (2.3) shows that (B) implies (A). Suppose that (C) holds. A standard argument (cf. [**1**, p. 49–50]) shows that there is a nonnegative number K for which $|\delta(a + b) - \delta(b)| \leqslant K\|a\|$ for all a and b in A. Hence $\delta(a + b) \leqslant K\|a\| + \delta(b)$, so Lemma (2.3) implies that all a belong to $Z(A)$, and the proof is complete.

Suppose that a has one-point spectrum in the Banach algebra A. Then the spectrum, and hence the spectral diameter, is continuous at a [**7**; **1**, Corollary 7, p. 8]. The following theorem shows that $\delta(x)$ cannot approach $\delta(a)$ too rapidly unless a is a scalar plus an element of the radical. We omit the proof since the theorem follows from Theorem (1.1) in the same way that the analogous result for the spectral radius [**2**, Theorem 4.1, p. 176] follows from Corollary (1.3).

THEOREM (2.5). *Suppose that a is an element with one-point spectrum in the Banach algebra with identity A. Then a is of the form $\lambda + r$ with r in $\mathrm{rad}(A)$ if and only if there is a number K for which $|\delta(a + h) - \delta(a)| \leqslant K\|h\|$ for all h in A.*

Note added in proof. After reading a preprint of this paper, Bernard Aupetit discovered a way to shorten the proof of Theorem (1.1) by using the fact that the spectral diameter is subharmonic.

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