ON A CONJECTURE OF M. S. ROBERTSON

ABDALLAH LYZZAIK

Abstract. We prove that two classes of univalent functions are equal. This settles a conjecture of M. S. Robertson in the affirmative.

1. Introduction. Recently, M. S. Robertson [3] introduced two classes of univalent functions, \( \mathcal{G} \) and \( \mathcal{G}^* \), and conjectured that they are equal. In this short note, we prove this conjecture.

First, let us define the classes \( \mathcal{G} \) and \( \mathcal{G}^* \).

Definition 1. Let \( \mathcal{G} \) be the class of all functions \( f \), regular and nonvanishing in \( B = \{ z : |z| < 1 \} \), with \( f(0) = 1 \), such that

\[
\Re \left( \frac{2z f'(z)}{f(z)} + \frac{1 + z}{1 - z} \right) > 0 \quad \text{for} \quad z \in B.
\]

Note that \( 1 \in \mathcal{G} \).

Let \( D \) be a domain, and let \( a \) belong to the closure of \( D \). We say that \( D \) is starlike with respect to \( a \) if for each \( z \in D \), every point \( tz + (1 - t)a \), with \( 0 < t < 1 \), belongs to \( D \).

Definition 2. Let \( \mathcal{G}^* \) be the class of functions \( f \), regular and univalent in \( B \), with \( f(0) = 1 \) and \( \lim_{r \to 1^-} f(r) = 0 \), such that \( f(B) \) is starlike with respect to the origin, and \( \Re(e^{i\alpha} f) > 0 \) for some real number \( \alpha \). Also, let \( 1 \in \mathcal{G}^* \).

For the sake of clarity, we remind the reader of some familiar definitions which are needed.

Definition 3. Let \( S^* \) be the class of all functions \( f \), regular in \( B \), with \( f(0) = 0 \), such that

\[
\Re z \frac{f'(z)}{f(z)} > 0 \quad \text{for} \quad z \in B.
\]

Note that there is no restriction on \( f'(0) \) in this definition.

It is known that each \( f \in S^* \) is univalent and maps \( B \) onto a domain starlike with respect to the origin.

Definition 4. Let \( S_g \) be the class of all functions \( f \) which satisfy one of the following conditions:

(a) \( f \) is regular and univalent in \( B \), and maps \( B \) onto a domain which contains the origin and is starlike with respect to the origin.

(b) \( f \) is of the form

\[
f(z) = h(z) \frac{(z - \zeta)(1 - \overline{\zeta} z)}{z}, \quad |\zeta| < 1,
\]

where \( h \in S^* \).

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The equivalence of conditions (a) and (b) was shown by J. Hummel [1]. This will be our main tool for the proof.

2. Proof of the conjecture.

THEOREM. \( \mathcal{G}^* = \mathcal{G} \).

Before we give the proof, we remark that the set-inclusion \( \mathcal{G} \subset \mathcal{G}^* \) was established by M. Robertson [3]. A short proof of this fact will be given as a part of the proof of the theorem.

PROOF OF THEOREM. (a) \( \mathcal{G}^* \subset \mathcal{G} \). Suppose that \( f \in \mathcal{G}^* \) is not identical to 1. It follows directly from Definition 2 that \( f^2 \) is univalent, \( \lim_{r \to 1^-} f(r) = 0 \), and \( f \) maps \( B \) onto a domain starlike with respect to the origin. Let \( D_n \) be the domain obtained from the union of the range of \( f^2 \) and the open disc centered at the origin and of radius \( 1/n \). Evidently, each \( D_n \) is simply connected. Let \( f_n \) be a conformal map from \( B \) onto \( D_n \) that satisfies \( f_n(0) = 1 \) and \( \arg f_n'(0) = \arg(f^2)'(0) \). By the Carathéodory Kernel Theorem [2, p. 29], \( f_n \to f^2 \) uniformly on compact subsets of \( B \). From Definition 4, each \( f_n \in S_g \). Hence for all \( n \) we can write

\[
f_n(z) = h_n(z)\frac{(z - z_n)(1 - \bar{z}_nz)}{z}, \quad |z_n| < 1,
\]

where \( h_n \in S^* \). It can be verified that

\[
f_n'(0) = \frac{1}{2} \frac{h_n''(0)}{h_n'(0)} + (1 + |z_n|^2)h_n'(0).
\]

Since \( f_n'(0) \to (f^2)'(0) \neq 0 \) and \( h \in S^* \) gives \( |h''(0)/h_n'(0)| \leq 4, h_n(0) \) is uniformly bounded for all \( n \). Hence, there exists a sequence of positive integers \( (n_k) \) so that \( (h_{n_k}) \) converges uniformly on compact subsets to either \( h \in S^* \) or zero. The latter case is impossible, otherwise \( zf_{n_k} \to 0 \) uniformly on compact subsets of \( B \) and \( f \) will be identical to zero. Suppose that \( z_{n_k} \to z, \) with \( |z| \leq 1 \) (otherwise we choose a subsequence of \( (z_{n_k}) \) that does so). Then we can write

\[
f^2(z) = h(z)\frac{(z - z)(1 - \bar{z}z)}{z}, \quad |z| \leq 1.
\]

Since \( f \) does not admit zero in \( \partial B, |z| = 1 \). Furthermore, since \( \lim_{r \to 1^-} f(r) = 0 \) and \( h \) is bounded away from zero for values of \( z \) close to \( \partial B \), \( z = 1 \). Therefore,

\[
f^2(z) = \frac{h(z)(1 - z)^2}{z},
\]

which yields

\[
\text{Re}\left\{ 2z\frac{f'(z)}{f(z)} + \frac{1 + z}{1 - z} \right\} = \text{Re}\left\{ \frac{h'(z)}{h(z)} \right\} > 0,
\]

and \( f \in \mathcal{G} \).

(b) \( \mathcal{G} \subset \mathcal{G}^* \). Let \( f \in \mathcal{G} \), with \( f \) not identical to 1, and let \( h(z) = f^2(z)z/(1 - z)^2 \). Then by simple calculation we have

\[
\text{Re}\left\{ \frac{z}{h(z)} \right\} = \text{Re}\left\{ 2zf'(z) + \frac{1 + z}{1 - z} \right\} > 0.
\]

So we have

\[
f^2(z) = h(z)\frac{(1 - z)^2}{z},
\]

where \( h \in S^*, \) with \( h'(0) = 1 \).
For every positive integer \( n \), let \( r_n = 1 - 1/n \), and let

\[
g_n(z) = -\frac{h(z)}{r_n} \frac{(z - r_n)(1 - r_nz)}{z}.
\]

From Definition 4, each \( g_n \in S_g \). Note that each \( g_n(0) = 1 \), and

\[
g_n'(0) = \frac{h''(0)}{2} - \frac{r_n^2 + 1}{r_n} \frac{h''(0)}{2} = 2 - (f^2)'(0) \neq 0, \infty,
\]

since \( h(z) \neq z/(1 - z)^2 \); otherwise \( f \) is identically 1. Also, note that \( g_n \to f^2 \) uniformly on compact subsets of \( B \). By Hurwitz’s Theorem, this implies the univalence of \( f^2 \) in \( B \). Let \( \Delta = f^2(B) \), and let \( \Delta_n = g_n(B) \). Then by the Carathéodory Kernel Theorem [2, p. 29], \( \Delta_n \to \Delta \) as \( n \to \infty \). Now we show that \( \Delta \) is a domain starlike with respect to the origin. Let \( w \in \Delta \). From the definition of the kernel [2, p. 28], there exists a domain \( U \) and a positive integer \( N \) such that \( 1, w \in U \) and \( U \) is contained in \( \Delta_n \) for all \( n > N \). Let \( H \) be the domain consisting of all open-closed segments starting from the origin and ending in \( U \). Since each \( g_n \in S_g \), each \( \Delta_n \) is starlike with respect to the origin. This implies that \( H \) is contained in \( \Delta_n \) for all \( n > N \). Hence \( H \) is contained in \( \Delta \), and consequently \( \Delta \) is starlike with respect to the origin. Since \( 0 \notin \Delta \), it is not hard to show that there exists a radial slit from the origin to infinity which does not meet \( \Delta \). Hence, the univalence of \( f^2 \), and the starlikeness of \( \Delta \) about the origin lead to the univalence of \( f \), and to the starlikeness of \( f(B) \) about the origin; moreover, there exists a real number \( a \) such that \( \Re(e^{ia}f) > 0 \) in \( B \). Since the origin is an accessible boundary point of \( f(B) \), there exists \( \gamma \), with \( |\gamma| = 1 \), so that \( \lim_{r \to 1} f(r\gamma) = 0 \) (see [2, p. 277]). Since \( h \) is bounded away from zero for values of \( z \) near \( \partial B \), (1) implies that \( \gamma = 1 \). Therefore, \( f \in g^* \) and the proof is complete.

The author can give an alternative shorter proof to the theorem based on D. Styer [4]. However, this proof is quite involved, and was avoided for the sake of clarity.

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Department of Mathematical Sciences, University of Petroleum and Minerals, Dhahran, Saudi Arabia