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A COMPLEMENT THEOREM FOR SHAPE CONCORDANT COMPACTA

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ABSTRACT. Let X and Y be compacta of polyhedral shape lying in the manifold M. Under suitable conditions, it is shown that if X and Y are shape concordant, then $M - X$ is homeomorphic to $M - Y$.

1. Introduction. A well-known fact in piecewise linear topology is Hudson's result [3] that "concordance implies isotopy" for codimension 3 embeddings of polyhedra in manifolds. Here we give a shape-theoretic version of this proposition, noting of course that the natural analog in the shape category of an unknotting theorem is a complement theorem. (The reader is referred to [8] for a good introduction to the relationship between shape-theoretic results and results in the piecewise linear and topological categories.) Our "shape concordance implies homeomorphic complement" result appears in §4 as Theorem 4.1.

The necessary groundwork for the statement and proof of Theorem 4.1 is laid in §§2 and 3. In these two sections we discuss the inessential loops condition for shape concordances (§2) and some crucial unpublished work of Stallings (§3) on which our techniques are largely based.

We assume that the reader is familiar with the fundamentals of shape theory [1] and piecewise linear (henceforth shortened to PL) topology [2], and our notations will be those commonly used in these areas.

2. Shape concordance and the inessential loops condition. We begin by defining the shape-theoretic counterpart of a concordance [3]; cf. Stallings' notion of "equivalence" in the following section.

(2.1) DEFINITION. Suppose X and Y are compacta in the interior of the manifold M and $\underline{f}\colon X \to Y$ is a shape equivalence. According to Husch and Ivansić [4], X is *shape concordant to Y relative to $\underline{f}$* if there exists a compactum $L \subset M \times I$ such that

(i) $L \cap (M \times \{0\}) = X \times \{0\}$ and $L \cap (M \times \{1\}) = Y \times \{1\}$,

(ii) the inclusions $X \times \{0\} \hookrightarrow L$ and $Y \times \{1\} \hookrightarrow L$ are shape equivalences, and

(iii) the diagram

$$\begin{tikzcd}[column sep=small] X \times \{0\} \arrow[rr, "\underline{f} \times \mathrm{const}"] \arrow[dr, hook] & & Y \times \{1\} \arrow[dl, hook'] \\ & L & \end{tikzcd}$$

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is commutative in the shape category. When using this concept, we shall not need to be explicit about the shape equivalence f, and so for us the statement that "X is shape concordant to Y" shall simply mean that there exists a compactum L for which (i) and (ii) above hold. We shall refer to L as a shape concordance *between* X and Y.

Our goal, of course, is to show that under suitable "niceness" conditions shape concordant compacta in a PL manifold have homeomorphic complements. These sorts of conditions are discussed in §2 of **[6]**, where the reader's attention is particularly directed to the notions of *fundamental dimension* and the *inessential loops condition* (ILC). Before proceeding to the main theorem, we will need a couple of technical results involving these concepts. For completeness, we begin with the definitions.

(2.2) DEFINITION. If X is a compactum, the *fundamental dimension* of X, denoted $\mathrm{Fd}(X)$, is defined to be $\min\{\dim Y: \mathrm{Sh}(X) = \mathrm{Sh}(Y), Y \text{ a compactum}\}$.

(2.3) DEFINITION. Suppose X is a compact subset of the manifold N. Then X satisfies the *inessential loops condition* (ILC) if for each neighborhood U of X, there exists a neighborhood V of X such that each loop in $V - X$ which is nullhomotopic in V is nullhomotopic in $U - X$.

We remark that in Definition 2.3, we do *not* require that X lie in the interior of N.

Now suppose X is a compact subset of the interior of the PL n-manifold M and U is an open neighborhood of X. According to Lemma 1.11 of **[8]**, if $\mathrm{Fd}(X) = k \leqslant n - 2$, then $\pi_i(U, U - X) = 0$. Moreover, if X satisfies ILC, then it follows from Lemma 1.12 of **[8]** that $\pi_i(U, U - X) = 0$ for $0 \leqslant i \leqslant n - k - 1$. Thus, if $k \leqslant n - 3$ and X satisfies ILC, $\pi_2(U, U - X) = 0 = \pi_1(U, U - X)$, and from the exactness of the homotopy sequence we immediately obtain the following.

(2.4) OBSERVATION. *If X is a compact subset of the interior of the PL n-manifold M, U is an open neighborhood of X,* $\mathrm{Fd}(X) \leqslant n - 3$, *and X satisfies ILC, then the inclusion-induced homomorphism $\pi_1(U - X) \to \pi_1(U)$ is an isomorphism. Thus each loop in $U - X$ which is nullhomotopic in U is nullhomotopic in $U - X$.*

The above observation was pointed out to me some time ago by Jerzy Dydak, who showed me a simple geometric argument verifying it even for the case $\mathrm{Fd}(X) \leqslant n - 2$. Since ILC is known not to be a good "niceness" condition to consider for co-fundamental-dimension 2 embeddings (it not being at all clear that there even exist any ILC embeddings in co-fundamental-dimension 2), this improvement does not seem to have great utility, and so we shall restrict our attention throughout this paper to embeddings whose co-fundamental-dimension is at least 3.

By repeatedly applying the Homotopy Extension Theorem for maps into ANR's and Lemma 1.12 of **[8]**, we obtain the following generalization of Observation 2.4. This result will be useful in proving Proposition 2.7.

(2.5) GENERAL POSITION LEMMA. *Suppose X is a compact subset of the interior of the PL n-manifold M, U is an open neighborhood of X,* $\mathrm{Fd}(X) \leqslant k$, *and X satisfies ILC. If P is a polyhedron, Q is a subpolyhedron of P, R is a subpolyhedron of P of dimension at most $n - k - 1$, and $f: P \to U$ is a map such that $f(Q) \cap X = \varnothing$, then f is homotopic* rel Q *to a map $g: P \to U$ such that $g(R) \cap X = \varnothing$.*

We now define the concept which is at the heart of our main result.

(2.6) DEFINITION. Suppose X and Y are compacta in the interior of the manifold M. Then X and Y are said to be ILC *shape concordant* in M if X and Y each satisfy ILC in M and if there is a shape concordance Z between X and Y in $M \times I$ such that Z satisfies ILC in $M \times I$.

Ordinarily, ILC has only been dealt with for compacta in the interior of a manifold, while the continuum Z in Definition 2.5 meets the boundary of $M \times I$. However, in the case of interest to us, the following proposition applies.

(2.7) PROPOSITION. *Suppose M is a PL n-manifold with empty boundary, Z is a compactum in $M \times I$,* $\mathrm{Fd}(Z \cap (M \times \{0\})) \leqslant n - 3$, *and* $\mathrm{Fd}(Z \cap (M \times \{1\})) \leqslant n - 3$. *If Z satisfies ILC in $M \times I$, then Z satisfies ILC in $M \times R$. The converse holds if $Z \cap (M \times \{0\})$ satisfies ILC in $M \times \{0\}$ and $Z \cap (M \times \{1\})$ satisfies ILC in $M \times \{1\}$.*

PROOF. Suppose first that Z satisfies ILC in $M \times I$, and let U be a neighborhood of Z in $M \times R$. If $U_0 = U \cap (M \times I)$, then there exists a compact PL-manifold neighborhood V_0 of Z in $M \times I$ such that $V_0 \subset \mathrm{Int}(U)$ and each loop in $V_0 - Z$ which is nullhomotopic in V_0 is nullhomotopic in $U_0 - Z$. Let

$$V_L = \mathrm{proj}(V_0 \cap (M \times \{0\})) \quad \text{and} \quad V_U = \mathrm{proj}(V_0 \cap (M \times \{1\})),$$

where proj: $M \times I \to M$ is the projection onto the first factor. Choose $\varepsilon > 0$ such that

$$V = V_0 \cup (V_L \times [-\varepsilon, 0]) \cup (V_U \times [1, 1 + \varepsilon]) \subset U.$$

Then V is a neighborhood of Z in $M \times R$, and Z will be seen to satisfy ILC in $M \times R$ once it is shown that each loop in $V - Z$ which is nullhomotopic in V is nullhomotopic in $U - Z$.

Let $l\colon S^1 \to V - Z$ be a loop which is nullhomotopic in V. If

$$Z_L = \mathrm{proj}(Z \cap (M \times \{0\})) \quad \text{and} \quad Z_U = \mathrm{proj}(Z \cap (M \times \{1\})),$$

it follows from Lemma 1.11 of **[8]** that $\pi_1(V_L, V_L - Z_L) = 0 = \pi_1(V_U, V_U - Z_U)$. Hence,

$$\pi_1(V_L \times [-\varepsilon, 0], (V_L - Z_L) \times [-\varepsilon, 0]) = 0$$

and

$$\pi_1(V_U \times [1, 1 + \varepsilon], (V_U - Z_U) \times [1, 1 + \varepsilon]) = 0.$$

From this, it is easily seen that $l \simeq l'$ in $V - Z$, where

$$l'\colon S^1 \to V - (Z \cup (Z_L \times [-\varepsilon, 0]) \cup (Z_U \times [1, 1 + \varepsilon])).$$

Pushing along the product structure in $V_L \times [-\varepsilon, 0]$ and $V_U \times [1, 1 + \varepsilon]$, we see that $l' \simeq l''$ in $V - Z$, where $l''(S^1) \subset V_0$. Since $l \simeq 0$ in V, we have $l'' \simeq 0$ in V, and so $l'' \simeq 0$ in V_0. Thus $l'' \simeq 0$ in $U_0 - Z$, and it follows that $l \simeq 0$ in $U - Z$.

For the converse, assume that Z satisfies ILC in $M \times R$ and that (using the notation established in the first part of the proof) each of Z_L and Z_U satisfies ILC in M. Let U be an open neighborhood of Z in $M \times I$, and let $U_L = \mathrm{proj}(U \cap (M \times \{0\}))$ and $U_U = \mathrm{proj}(U \cap (M \times \{1\}))$. Let $U_0 = U \cup (U_L \times (-1, 0]) \cup (U_U \times [1, 2))$.

Then U_0 is a neighborhood of Z in $M \times R$, and so there exists a neighborhood V_0 of Z in $M \times R$ such that each loop in $V_0 - Z$ which is nullhomotopic in V_0 is nullhomotopic in $U_0 - Z$. Let $V = V_0 \cap (M \times I)$. Then V is a neighborhood of Z in $M \times I$, and Z will be seen to satisfy ILC in $M \times I$ once it is shown that each loop in $V - Z$ which is nullhomotopic in V is nullhomotopic in $U - Z$.

Let l: $S^1 \to V - Z$ be a loop which is nullhomotopic in V. We may assume that $l(S^1) \cap (M \times \{0\}) = \varnothing = l(S^1) \cap (M \times \{1\})$. Since $l \simeq 0$ in V_0, there is an extension l_*: $D^2 \to U_0 - Z$ of l. We may assume that l_* is PL and in general position with respect to $(M \times \{0\}) \cup (M \times \{1\})$, so that $l_*^{-1}((U_L \times (-1,0]) \cup (U_U \times [1,2)))$ is the union of finitely many pairwise disjoint punctured disks. Let E be one of these, which we assume maps by l_* into $U_L \times (-1,0]$. Pushing along the product structure, we may homotope $l_*|E$ rel ∂E to a map l'_*: $E \to U_L \times \{0\}$. Since Z_L satisfies ILC in M and Fd $Z_L \leqslant n - 3$, it follows from General Position Lemma 2.5 that l'_* is homotopic rel ∂E to a map l''_*: $E \to U_L \times \{0\}$ such that $l''_*(E) \cap (Z_L \times \{0\}) = \varnothing$. Pushing $l''_*(E)$ slightly away from $M \times \{0\}$, we find that l_* is homotopic rel S^1 to a map l_{**}: $D^2 \to U_0 - Z$, where l_{**} is PL and in general position with respect to $(M \times \{0\}) \cup (M \times \{1\})$, and

$$l_{**}^{-1}((U_L \times (-1,0]) \cup (U_U \times [1,2)))$$

has fewer components than $l_*^{-1}((U_L \times (-1,0]) \cup (U_U \times [1,2)))$. By induction, we finally obtain an extension of l which maps D^2 into $U - Z$, thereby completing the proof.

The following result will be used in the proof of Theorem 4.1.

(2.8) LEMMA. *Suppose Z is a compact subset of the manifold N. Let*

$$N^+ = N \cup_h (\partial N \times I),$$

where h: $\partial N \times \{0\} \to N$ is defined by $h(x,0) = x$ for all $x \in \partial N$, and let

$$Z^+ = Z \cup ((Z \cap \partial N) \times I) \subset N^+.$$

If Z satisfies ILC in N, then Z^+ satisfies ILC in N^+.

PROOF. Let U be a neighborhood of Z^+ in N^+. Let V be a neighborhood of Z^+ in N^+ such that

(1) $V \subset U$,

(2) $V = (V \cap N) \cup ((V \cap \partial N) \times I)$ and

(3) each loop in $(V \cap N) - Z$ which is nullhomotopic in $V \cap N$ is nullhomotopic in $(U \cap N) - Z$.

Let l: $S^1 \to V - Z^+$ be a loop which is nullhomotopic in V. Define F: $V \to V$ by

$$F(x) = \begin{cases} x & \text{if } x \in V \cap N, \\ y & \text{if } x = (y,t) \in (V \cap \partial N) \times I. \end{cases}$$

Then $l \simeq F \circ l$ in $V - Z^+$. Noting that $F \circ l(S^1) \subset (V \cap N) - Z$ and that $F \circ l$ is nullhomotopic in $V \cap N$, it follows that $F \circ l$ is nullhomotopic in $(U \cap N) - Z$. Hence l is nullhomotopic in $U - Z^+$.

3. Stallings' notion of "equivalence". Our main result will depend on some work of Stallings [7] which, while unpublished, seems to have become a part of the folklore. For completeness, we shall review the relevant portions of that work here.

(3.1) DEFINITION. Suppose K_0 and K_1 are polyhedra lying in the interior of the PL manifold M and α: $K_0 \to K_1$ is a simple homotopy equivalence. Then K_0 is *equivalent* to K_1 (relative to α) if there is a polyhedron $L \subset M \times I$ such that

(i) $L \cap (M \times \{0\}) = K_0 \times \{0\}$ and $L \cap (M \times \{1\}) = K_1 \times \{1\}$,

(ii) the inclusions $K_0 \times \{0\} \hookrightarrow L$ and $K_1 \times \{1\} \hookrightarrow L$ are simple homotopy equivalences, and

(iii) the diagram

$$\begin{tikzcd} K_0 \times \{0\} \arrow[rr, "\alpha \times \mathrm{const}"] \arrow[dr, hook] & & K_1 \times \{1\} \arrow[dl, hook'] \\ & L & \end{tikzcd}$$

commutes up to homotopy.

It is clear that this notion provided the motivation for the concept of shape concordance. As in the case of shape concordances, we shall not need to be explicit about the simple homotopy equivalence α, and we shall refer to a polyhedron L satisfying (i) and (ii) above as an *equivalence* between K_0 and K_1.

Stallings' paper contains a result which is quite analogous to the main theorem of [4]. For us, though, the more germane part of his work lies in what he calls the "Neighborhood-Isotopy Theorem". It appears as Theorem 5.3 of [7], and its proof involves a relatively straightforward application of the relative S-Cobordism Theorem. For completeness, we give the statement of it now.

(3.2) THEOREM. *Let K_0 and K_1 be k-dimensional polyhedra in the PL n-manifold M, $n \geqslant 6$ and $\partial M = \varnothing$, which are equivalent by a polyhedron L of dimension at most $k+1$. If $k \leqslant n-3$, then there is a PL homeomorphism* g: $M \times I \to M \times I$ *which is the identity on $M \times \{0\}$ and which takes $N_1 \times \{1\}$ onto $N_0 \times \{1\}$, where N_0 and N_1 are regular neighborhoods of K_0 and K_1 respectively.*

Recall that a *deleted product neighborhood* of the compactum X in the interior of the manifold M is a compact manifold neighborhood N of X such that $N - X \cong \partial N \times [0, 1)$. The proof of the following fact is an easy exercise in matching the fibers in a product.

(3.3) PROPOSITION. *Suppose X and Y are compacta lying in the interior of the manifold M and having a common deleted product neighborhood. Then $M - X \cong M - Y$.*

Since a regular neighborhood of a polyhedron lying in the interior of a manifold is a deleted product neighborhood, the following (which will be generalized by Theorem 4.1) is an immediate consequence of Theorem 3.2 and Proposition 3.3.

(3.4) COROLLARY. *Let K_0 and K_1 be k-dimensional polyhedra in the PL n-manifold M, $n \geqslant 6$ and $\partial M = \varnothing$, which are equivalent by a polyhedron L of dimension at most $k+1$. If $k \leqslant n-3$, then $M - K_1 \cong M - K_2$.*

We conclude this section by stating, for completeness, another of Stallings' results which will be used as a tool in §4. This result appears in [7] as Theorem 3.1.

(3.5) LEMMA. *Let $f\colon K \to M$ be an $(s+1)$-connected PL map of the polyhedron K into the PL manifold M and let $\Sigma \subset K$ contain the singular set of f, where* $\dim \Sigma \leqslant s$. *Then there exists a polyhedron L which contains $f(K)$, such that* (1) *the composition $K \to f(K) \hookrightarrow L$ is a simple homotopy equivalence*, (2) $\dim(L - f(K)) \leqslant s + 2$ *and* (3) *the inclusion $f(K) \hookrightarrow M$ extends to a map $g\colon L \to M$.*

4. A "concordance implies unknotting" theorem in the shape category. In this section the main result of the paper is stated and proved.

(4.1) THEOREM. *Suppose X and Y are compacta in the PL n-manifold M, $n \geqslant 6$ and $\partial M = \varnothing$, which are ILC shape concordant in M. Suppose further that X has the shape of a polyhedron K of dimension at most k, where $k \leqslant n-3$. Then $M - X \cong M - Y$.*

PROOF. Let $Z \subset M \times I$ be a shape concordance between X and Y in $M \times I$, where Z satisfies ILC in $M \times I$, $Z \cap (M \times \{0\}) = X \times \{0\}$, and $Z \cap (M \times \{1\}) = Y \times \{1\}$. If $i = 0, 1, 2, \ldots,$ let $Z_i = Z \cup (X \times [-i, 0]) \cup (Y \times [1, 1+i])$. Then according to Lemma 2.8, Z_i satisfies ILC in $M \times [-i, 1+i]$, and so by Proposition 2.7, Z_i satisfies ILC in $M \times R$. Since $\mathrm{Sh}(Z_i) = \mathrm{Sh}(Z) = \mathrm{Sh}(X \times \{0\}) = \mathrm{Sh}(K)$, it follows from Corollary 1 of [5] that Z_i has a deleted product neighborhood U_i lying in the interior of $M \times [-i-1, 2+i]$. In fact, for $i = 0, 1, \ldots, 2k+3-n$, we may choose such neighborhoods so that

$$\begin{aligned} U_{2k+3-n} &\supset \mathrm{Int}(U_{2k+3-n}) \supset U_{2k+2-n} \supset \mathrm{Int}(U_{2k+2-n}) \supset U_{2k+1-n} \\ &\supset \cdots \supset U_1 \supset \mathrm{Int}(U_1) \supset U_0. \end{aligned}$$

(Note: If $2k+3-n < 0$, then the above is not sensible, but it will follow easily from the proof below that it suffices in this case to simply deal with U_0 and U_1.) Now, if $0 < i \leqslant 2k+3-n$, note that the diagram

$$\begin{tikzcd} Z_i \arrow[r, hook] & U_i \\ Z_{i-1} \arrow[r, hook] \arrow[u, hook] & U_{i-1} \arrow[u, hook] \end{tikzcd}$$

commutes and that each of the left-most three of these inclusion maps generates a shape equivalence. It follows that the right-most map also does, and so the map $U_{i-1} \hookrightarrow U_i$ is a homotopy equivalence.

Let $f\colon K \to X \times \{0\}$ be a shape equivalence. By the proof of Corollary 1 of [5], there exists an at-most-k-dimensional polyhedron $K_0 \subset \mathrm{Int}\, M$, a simple homotopy equivalence $f\colon K \to K_0 \times \{0\}$, and a regular neighborhood N_0 of K_0 in M such that $X \times \{0\} \subset N_0 \times \{0\} \subset U_0$, N_0 is a deleted product neighborhood of X, the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[d, hook] & K \arrow[l, "f"'] \arrow[d, "f"] \\ N_0 \times \{0\} & K_0 \times \{0\} \arrow[l, hook'] \end{tikzcd}$$

commutes in the shape category and, if $0 < i \leqslant 2k + 3 - n$, $K_0 \times [-i, 0] \subset U_i$. Since N_0 is a deleted product neighborhood of X and of K_0, it follows from Proposition 3.3 that $M - X \cong M - K_0$.

Now, by hypothesis, the maps $X \times \{0\} \to Z$ and $Y \times \{1\} \to Z$ induce shape equivalences. Hence, there exists a shape equivalence $\underline{g}$: $X \times \{0\} \to Y \times \{1\}$ such that the diagram

$$\begin{tikzcd}
 & Y \times \{1\} \arrow[dl, hook'] \\
Z & \\
 & X \times \{0\} \arrow[ul, hook] \arrow[uu, "\underline{g}"']
\end{tikzcd}$$

commutes in the shape category. Noting that $\underline{gf}$: $K \to Y \times \{1\}$ is a shape equivalence we may find, as in the preceding paragraph, an at-most-k-dimensional polyhedron $K_1 \subset \operatorname{Int} M$, a simple homotopy equivalence h: $K \to K_1 \times \{1\}$, and a regular neighborhood N_1 of K_1 in M such that $Y \times \{1\} \subset N_1 \times \{1\} \subset U_0$, N_1 is a deleted product neighborhood of Y, the diagram

$$\begin{tikzcd}
N_1 \times \{1\} & K_1 \times \{1\} \arrow[l, hook'] \\
Y \times \{1\} \arrow[u, hook] & K \arrow[u, "h"'] \arrow[l, "\underline{gf}"]
\end{tikzcd}$$

commutes in the shape category and, if $0 < i \leqslant 2k + 3 - n$, $K_1 \times [1, 1 + i] \subset U_i$. Since N_1 is a deleted product neighborhood of Y and of K_1, it follows from Proposition 3.3 that $M - Y \cong M - K_1$.

To summarize, we have found that

(1) the diagram

$$\begin{tikzcd}
 & & N_1 \times \{1\} \arrow[dll, hook'] & & K_1 \times \{1\} \arrow[ll, hook'] \\
 & & Y \times \{1\} \arrow[u, hook] \arrow[dll, hook'] \arrow[dl, hook'] & & \\
U_0 & Z \arrow[l, hook'] & & K \arrow[ul, "\underline{gf}"'] \arrow[dl, "\underline{f}"] \arrow[uur, "h"'] \arrow[ddr, "f"] & \\
 & & X \times \{0\} \arrow[uu, "\underline{g}"] \arrow[ul, hook] \arrow[ull, hook] \arrow[d, hook] & & \\
 & & N_0 \times \{0\} \arrow[uull, hook] & & K_0 \times \{0\} \arrow[ll, hook']
\end{tikzcd}$$

commutes in the shape category, where each arrow indicates either a shape equivalence or a map which induces a shape equivalence, and

(2) to prove that $M - X \cong M - Y$, it suffices to show that $M - K_0 \cong M - K_1$. To prove this latter, it suffices to show that K_0 and K_1 are equivalent in the sense of Stallings by an equivalence of dimension at most $k + 1$, and the remainder of the proof will be devoted to establishing this.

Let g': $K_0 \times \{0\} \to K_1 \times \{1\}$ be the composition of h: $K \to K_1 \times \{1\}$ and a homotopy inverse of f. Then g' is a simple homotopy equivalence and, since the above diagram commutes in the shape category, we may add g' to it and see that g' is homotopic to $\mathrm{id}_{K_0 \times \{0\}}$ in U_0. It follows that if g: $K_0 \to K_1$ is defined by $(g(x), 1) = g'(x, 0)$ for $x \in K_0$, then there is a map F_0: $M(g) \to U_0$ such that $F_0(x, t) = (x, t)$ if $x \in K_0$ and $t = 0$ or if $x \in K_1$, and $t = 1$. (Here $M(g)$ denotes the mapping cylinder of g.)

Let $C_1 = M(g) \cup (K_0 \times [-1, 0]) \cup (K_1 \times [1, 2])$, and let F_1: $C_1 \to U_1$ be obtained by extending F_0 over $(K_0 \times [-1, 0]) \cup (K_1 \times [1, 2])$ via the identity. Note that the inclusions $K_0 \times \{-1\} \hookrightarrow C_1$ and $K_1 \times \{2\} \hookrightarrow C_1$ are simple homotopy equivalences. By slight adjustment rel$[(K_0 \times [-1, 0]) \cup (K_1 \times [1, 2])]$ we may assume F_1 to be piecewise linear and in general position, with

$$F_1\left(C_1 - \left[(K_0 \times \{-1\}) \cup (K_1 \times \{2\})\right]\right) \subset U_1 \cap (M \times (-1, 2)).$$

Observe that the diagram

$$\begin{tikzcd}
C_1 \arrow[drr, "F_1"] & & \\
K_1 \times \{2\} \arrow[u, hook, "\simeq"] \arrow[rr, hook] & & U_1 \\
K_1 \times \{1\} \arrow[u, "\mathrm{id} \times \mathrm{const}", "\simeq"'] \arrow[rr, hook, "\cong"'] & & U_0 \arrow[u, hook, "\simeq"']
\end{tikzcd}$$

commutes up to homotopy and that the maps marked " $\simeq$ " are homotopy equivalences. It follows that F_1 is a homotopy equivalence, and in particular that F_1 is $(2k + 2 - n)$-connected, while the singular set of F_1 has dimension $\leqslant 2k + 1 - n$. By Lemma 3.5, there exists a polyhedron L_1 which contains $F_1(C_1)$, such that the composition $C_1 \to F_1(C_1) \hookrightarrow L_1$ is a simple homotopy equivalence,

$$\dim(L_1 - F_1(C_1)) \leqslant 2k + 3 - n,$$

and there exists a PL map G_1: $L_1 \to U_1$ which extends the inclusion $F_1(C_1) \hookrightarrow U_1$. Note that the inclusions $K_0 \times \{-1\} \hookrightarrow L_1$ and $K_1 \times \{2\} \hookrightarrow L_1$ are simple homotopy equivalences.

Let $C_2 = L_1 \cup (K_0 \times [-2, -1]) \cup (K_1 \times [2, 3])$, and let F_2: $C_2 \to U_2$ be obtained by extending G_1 over $(K_0 \times [-2, -1]) \cup (K_1 \times [2, 3])$ via the identity. Note that the inclusions $K_0 \times \{-2\} \to C_2$ and $K_1 \times \{3\} \to C_2$ are simple homotopy equivalences.

By slight adjustment rel$[F_1(C_1) \cup (K_0 \times [-2,-1]) \cup (K_1 \times [2,3])]$, we may assume F_2 to be PL and in general position, with

$$F_2\big(C_2 - [(K_0 \times \{-2\}) \cup (K_1 \times \{3\})]\big) \subset U_2 \cap (M \times (-2,3)).$$

Observe that the diagram

$$\begin{tikzcd} C_2 \arrow[drr, "F_2"] & & \\ K_1 \times \{3\} \arrow[u, hook, "\simeq"] \arrow[rr, hook] & & U_2 \\ K_1 \times \{2\} \arrow[u, "\simeq", "\mathrm{id} \times \mathrm{const}"'] \arrow[rr, hook, "\simeq"'] & & U_2 \arrow[u, hook, "\simeq"'] \end{tikzcd}$$

commutes up to homotopy and that the maps marked "$\simeq$" are homotopy equivalences. It follows that F_2 is a homotopy equivalence, and in particular that F_2 is $(2k+1-n)$-connected, while the singular set of F_2 has dimension $\leqslant (k+1) + (2k+3-n) - (n+1) = (2k-n) + (k+3-n) \leqslant 2k-n$.

Note that the upper bound for the dimension of the singular set of F_2 is less (by 1) than that of F_1. We continue the above process inductively, finally obtaining an at-most-$(k+1)$-dimensional polyhedron C_{2k+3-n} containing $(K_0 \times \{-2k-3+n\}) \cup (K_1 \times \{2k+4-n\})$ such that

$$K_0 \times \{-2k-3+n\} \hookrightarrow C_{2k+3-n} \quad \text{and} \quad K_1 \times \{2k+4-n\} \hookrightarrow C_{2k+3-n}$$

are simple homotopy equivalences, and a PL map $F_{2k+3-n}\colon C_{2k+3-n} \to U_{2k+3-n}$ such that the singular set of F_{2k+3-n} has dimension $\leqslant -1$ (i.e. such that F_{2k+3-n} is an embedding) and such that F_{2k+3-n} agrees with the identity on

$$(K_0 \times \{-2k-3+n\}) \cup (K_1 \times \{2k+4-n\}).$$

This clearly shows that K_0 and K_1 are equivalent in the sense of Stallings by an equivalence of dimension at most $k+1$.

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