

A TOWER OF SPECTRA THAT REALIZES A CHAIN COMPLEX

PEDRO A. SUAREZ, S. J.

ABSTRACT. This paper presents the construction of a tower of spectra Y_j with k -invariants coming from the relations $Sq^1(XSq^{2^{j-1}})Sq^1(XSq^{2^j}) = 0$ in $A/A Sq^1$, for $0 \leq j \leq 5$ and $A =$ Steenrod algebra mod 2, such that Y_5 has prescribed homotopy groups: $\pi_n(Y_5) = Z$ (integers) if $n = 2^{j+1} - 2$, and zero otherwise.

I. Introduction. This paper presents an example of a tower of spectra associated with a chain complex over the Steenrod algebra module $A/A Sq^1$ (A is the mod 2 Steenrod algebra).

A problem which is related to our present task was posed by F. Peterson in 1970 [2]. Peterson asks if there exists an Ω -spectrum $\mathbf{X}(n)$ with homotopy groups

$$\pi_k(\mathbf{X}(n)) = \begin{cases} Z_2 & \text{if } k = 2^r n, r \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and such that the r th stage Postnikov system has k -invariants coming from the relations

$$\begin{aligned} Sq^{n+1} &= 0, \\ Sq^{2n+1} Sq^{n+1} &= 0, \\ &\dots \\ Sq^{2^{r-1}n+1}(\dots Sq^{n+1}) &= 0. \end{aligned}$$

Furthermore, the 0th term of the spectrum should be a product of $K(Z_2, n) \times K(Z_2, 2^n) \times \dots$ with twisted Hopf algebra structure, so that the homology generator in dimension n generates a polynomial algebra.

By using an algebraic argument and the work of Segal [3], R. Steiner gives a positive answer to this question [4].

One can consider an extension of Peterson's problem and ask if there exists a spectrum (or a tower of spectra) that realizes a relation $a_{n+1}a_n = 0$, where a_n may be perhaps more complicated than $Sq^{2n+1} Sq^{n+1} = 0$. A simple case is the spectrum bu , whose tower realization is associated with the relation $Sq^3 Sq^3 = 0$ in $A/A Sq^1$. In

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this paper we will construct a tower of spectra with k -invariants coming from the relation $Sq^1(X Sq^{2^{k+1}})Sq^1(X Sq^{2^k}) = 0$ in $A/A Sq^1$, where $X: A \rightarrow A$ is the canonical Thom antiautomorphism.

We will work in the category of spectra localized at two and homotopy classes of maps between such spectra. $KG(m)$ will represent the m -fold suspension of the Eilenberg-Mac Lane spectrum KG (or $KG(0)$) for a group G . Thus, $KG(m) = \Sigma^m KG(0)$, and $(KG(m))_n = K(G, m + n)$ for all nonnegative integers m, n . bu is the Ω -spectrum representing connective complex K -theory. The cohomology groups with coefficients in Z_2 of these spectra are well known [5].

II. Statement of results. First, we wish to state

PROPOSITION 1. *Let $D_k = A/A Sq^1$ and $d_k = Sq^1(X Sq^{2^k})$ for $k \geq 1$. Then*

$$\mathcal{C}: D_1 \xleftarrow{d_1} D_2 \xleftarrow{d_2} D_3 \xleftarrow{\dots} D_k \xleftarrow{d_k} D_{k+1} \xleftarrow{d_{k+1}} \dots$$

is a chain complex.

The proof of this proposition rests on

LEMMA 2. *The following relations hold in $A/A Sq^1$ for $k \geq 1$*

- (a) $Sq^1(X Sq^{2^k}) = Sq^{2^{k-1}} Sq^1(X Sq^{2^{k-1}})$;
- (b) $(X Sq^{2^k}) Sq^1(X Sq^{2^{k+1}}) = 0$;
- (c) $Sq^1(X Sq^{2^{k+1}}) Sq^1(X Sq^{2^k}) = 0$.

Finally, the main result is

THEOREM 3. *The following tower of fibrations of spectra realizes a finite portion of \mathcal{C} (for $1 \leq k \leq 6$):*

KZ(62)	\rightarrow	Y_5		
	j_5			
		\downarrow		
KZ(30)	\rightarrow	Y_4	\rightarrow	KZ(63)
	j_4		u_4	
		\downarrow		
KZ(14)	\rightarrow	Y_3	\rightarrow	KZ(31)
	j_3		u_3	
		\downarrow		
KZ(6)	\rightarrow	Y_2	\rightarrow	KZ(15)
	j_2		u_2	
		\downarrow		
KZ(2)	\rightarrow	Y_1	\rightarrow	KZ(7)
	j_1		u_1	
		\downarrow		
		$KZ(0) = Y_0$	\rightarrow	KZ(3)
			u_0	

where

$$\begin{aligned}
 u_0 &= b \text{Sq}^2(i_0) \quad (b = \text{Bockstein homomorphism}), \\
 u_1 &= \text{Sq}^2(v_5), \quad \text{with } v_5 = \text{nonzero generator of } H^5(\Sigma^5bu; Z), \\
 u_2 &= \text{Sq}^4 \text{Sq}^2 \text{Sq}^1(v_8) = (X \text{Sq}^7)v_8 \quad \text{and } v_8 \in H^8(Y_2; Z_2), \\
 u_3 &= \text{Sq}^8 \text{Sq}^4 \text{Sq}^1(v_{18}) = (X \text{Sq}^{13})v_{18} \quad \text{and } v_{18} \in H^{18}(Y_3; Z_2), \\
 u_4 &= \text{Sq}^{16} \text{Sq}^8 \text{Sq}^1(v_{38}) \quad \text{and } v_{38} \in H^{38}(Y_4; Z_2).
 \end{aligned}$$

Furthermore, the homotopy groups of the spectrum Y_5 are

$$\pi_i(Y_5) = \begin{cases} Z & \text{if } i = 2^k - 2, 1 \leq k \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

III. Proofs of 1, 2 and 3. The proof of Proposition 1 clearly rests on the assertion of Lemma 2(c). Lemma 2(a) is proved by mathematical induction and the use of the relation

$$X(\text{Sq}^{2^n-k}) = \text{Sq}^{2^{n-1}} \text{Sq}^{2^{n-2}} \dots \text{Sq}^{2^{k-1}} (X \text{Sq}^{2^{k-1}-k})$$

for any integers n, k such that $n \geq k \geq 1$ (see [1]). 2(b) is proved by using 2(a) and the Adem relation $\text{Sq}^{2^n-1} \text{Sq}^n = 0$. 2(c) follows immediately from (b).

The proof of Theorem 3 involves the construction of a Postnikov tower (the T -tower) and then an induced tower of spectra Y_k ($1 \leq k \leq 6$). The final picture looks like this:

$$\begin{array}{ccccccc}
 KZ(62) & \xrightarrow{j_5} & Y_5 & \xrightarrow{a_5} & T_5 & & \\
 & & \downarrow & & \downarrow & & \\
 KZ(30) & \xrightarrow{j_4} & Y_4 & \xrightarrow{a_4} & T_4 & \xrightarrow{f_4} & KZ(63) \\
 & & \downarrow & & \downarrow & & \\
 KZ(14) & \xrightarrow{j_3} & Y_3 & \xrightarrow{a_3} & T_3 & \xrightarrow{f_3} & KZ(31) \\
 & & \downarrow & & \downarrow & & \\
 KZ(6) & \xrightarrow{j_2} & Y_2 & \xrightarrow{a_2} & T_2 & \xrightarrow{f_2} & KZ(15) \\
 & & \downarrow & & \downarrow & & \\
 KZ(2) & \xrightarrow{j_1} & Y_1 & \xrightarrow{a_1} & T_1 & \xrightarrow{f_1} & KZ(7) \\
 & & \downarrow & & \downarrow & & \\
 KZ(0) & \xrightarrow{j_0=\text{Id}} & Y_0 & \xrightarrow{a_0} & \Sigma^3bu = T_0 & \xrightarrow{f_0} & KZ(3)
 \end{array}$$

The following is a sketch of the construction of this double tower of spectra. A more detailed discussion can be found in [6].

1. $f_0 \in H^3(\Sigma^3bu; Z) = [\Sigma^3bu, KZ(3)]$ corresponds to the generator 1.

2. Each T_{i+1} is induced by f_i over the path fibration of the Eilenberg-Mac Lane spectrum $KZ(2^{i+2} - 1)$ for $0 \leq i \leq 4$.

3. $a_0 \in [KZ(0), \Sigma^3bu]$ is such that $f_0a_0 = bSq^2(i_0) \in [KZ(0), KZ(3)]$, where b is the Bockstein homomorphism in cohomology and i_0 is the bottom class in $H^0(KZ(0); Z)$.

4. $T_1 \simeq \Sigma^5bu$. This can be seen by considering the sequence of fibrations $\Sigma^{n+3}bu \rightarrow \Sigma^{n+1}bu \rightarrow KZ(n+1)$, obtained by Bott periodicity.

5. The T -tower is constructed as follows:

$$\begin{aligned}
 f_0 &= \text{nonzero generator of } H^3(\Sigma^3bu; Z), \\
 f_1 &= Sq^2(v_5) \in H^7(\Sigma^5bu; Z), \text{ where } v_5 = \text{nonzero generator of } H^5(\Sigma^5bu; Z), \\
 f_2 &= Sq^4Sq^2Sq^1(v_8) \in H^{15}(T_2; Z), \\
 f_3 &= Sq^8Sq^4Sq^1(v_{18}) \in H^{31}(T_3; Z), \\
 f_4 &= Sq^{16}Sq^8Sq^1(v_{38}) \in H^{63}(T_4; Z),
 \end{aligned}$$

where each v_m appears by exactness of the long exact sequence in cohomology of each fibration ($m = 2^{k+1} + 2^{k-1} - 2; k = 2, 3, 4$) and each f_k is an integer class ($0 \leq k \leq 4$).

6. The Y -tower is induced over the T -tower in the following way. At the first stage we have

$$\begin{array}{ccccc}
 & & & & KZ(2) \\
 & & & \swarrow & \downarrow & \searrow \\
 Y_1 & \xrightarrow{a_1} & \Sigma^5bu & \rightarrow & PKZ(3) \\
 \downarrow & & \downarrow & & \downarrow \\
 KZ(0) & \xrightarrow{a_0} & \Sigma^3bu & \rightarrow & KZ(3)
 \end{array}$$

where $a_0 \in [KZ(0), \Sigma^3bu]$ such that

$$f_0a_0 = bSq^2(i_0) \in [KZ(0), KZ(3)] \cong H^3(KZ(0); Z).$$

Then Y_1 is the total spectrum over the path fibration $KZ(2) \hookrightarrow PKZ(3) \rightarrow KZ(3)$ induced by f_0a_0 . By the functorial properties of induced fibrations, Y_1 is equivalent to the total spectrum induced by a_0 over the fibration $KZ(2) \hookrightarrow \Sigma^5bu \rightarrow \Sigma^3bu$, so there exists a map $a_1: Y_1 \rightarrow \Sigma^5bu$.

Taking the composition of a_1 with $f_1 \in [T_1, KZ(7)]$ we obtain a map $f_1a_1 \in [Y_1, KZ(7)]$ which induces the spectrum Y_2 over the path fibration $KZ(6) \hookrightarrow PKZ(7) \rightarrow KZ(7)$, and the process can be repeated again, obtaining $u_k = f_k a_k$ ($0 \leq k \leq 4$).

7. The following diagram illustrates how cohomology classes are mapped at each level of the tower in their respective long exact cohomology sequences (the rings on the right indicate the coefficients in cohomology and τ is transgression):

$k = 0$		i_2	$\xrightarrow{\tau}$	$b \text{Sq}^2(i_0)$	(Z)
		i_2	\mapsto	$\text{Sq}^3(i_0)$	(Z_2)
	v_5	$\xrightarrow{j_1^*}$	$\text{Sq}^3 i_2$	\mapsto	0
	$\text{Sq}^2(v_5)$	\mapsto	$\text{Sq}^1(X \text{Sq}^4) i_2$		(Z_2)
$k = 1$		i_6	$\xrightarrow{\tau}$	$\text{Sq}^2(v_5)$	(Z)
	v_8	$\xrightarrow{j_2^*}$	$\text{Sq}^2 i_6$	\mapsto	0
	$(X \text{Sq}^7) v_8$	\mapsto	$\text{Sq}^1(X \text{Sq}^8) i_6$		(Z_2)
$k = 2$		i_{14}	$\xrightarrow{\tau}$	$(X \text{Sq}^7) v_8$	(Z)
	v_{18}	$\xrightarrow{j_3^*}$	$(X \text{Sq}^4) i_{14}$	\mapsto	0
	$(X \text{Sq}^{13}) v_{18}$	\mapsto	$\text{Sq}^1(X \text{Sq}^{16}) i_{14}$		(Z_2)
$k = 3$		i_{30}	$\xrightarrow{\tau}$	$X \text{Sq}^{13}(v_{18})$	(Z)
	v_{38}	$\xrightarrow{j_4^*}$	$(X \text{Sq}^8) i_{30}$	\mapsto	0
	$\text{Sq}^{16} \text{Sq}^8 \text{Sq}^1(v_{38})$	\mapsto	$\text{Sq}^1(X \text{Sq}^{32}) i_{30}$		(Z_2)
$k = 4$		i_{62}	$\xrightarrow{\tau}$	$\text{Sq}^{16} \text{Sq}^8 \text{Sq}^1(v_{38})$	(Z)
	v_{78}	$\xrightarrow{j_5^*}$	$(X \text{Sq}^{16}) i_{62}$	\mapsto	0
	$\text{Sq}^{32} \text{Sq}^{16} \text{Sq}^1 v_{78}$	\mapsto	$\text{Sq}^1(X \text{Sq}^{64}) i_{62}$		(Z_2)

8. It remains to show that this tower realizes (or is associated with) the chain complex \mathcal{C} . By Lemma 2,

$$d_k d_{k+1} = \text{Sq}^1(X \text{Sq}^{2^{k+1}}) \text{Sq}^1(X \text{Sq}^{2^k}) = 0 \pmod{A/A \text{Sq}^1}$$

for all nonnegative integers k . But we also have for $1 \leq k \leq 5$:

$$0 = \text{Sq}^1 X \text{Sq}^{2^{k+1}} \text{Sq}^1 X \text{Sq}^{2^k}(i_{2^{k-2}}) = \text{Sq}^1 X \text{Sq}^{2^{k+1}} j_{k-1}^*(\text{Sq}^{2^{k-1}} \text{Sq}^{2^{k-2}} \text{Sq}^1(v_s)),$$

letting $s = 2^k + 2^{k-2} - 2$. We note that

$$\text{Sq}^{2^{k-1}} \text{Sq}^{2^{k-2}} \text{Sq}^1(v_s) = \tau(i_{2^{k+1}-2}).$$

Thus, we obtain

$$0 = \text{Sq}^1 X \text{Sq}^{2^{k+1}} j_{k-1}^* \tau(i_{2^{k+1}-2}) = j_{k-1}^* \tau \text{Sq}^1 X \text{Sq}^{2^{k+1}}(i_{2^{k+1}-2}).$$

The homotopy groups of the spectrum Y_5 are those stated in the theorem, as can be easily checked from the diagram, by the exact homotopy sequence at each stage.

Technical difficulties with this method preclude our going higher and obtaining an infinite tower. Concretely, it is not clear that a class $v_s \in H^t(Y_k; Z_2)$ exists ($s = 2^{k+1} + 2^{k-1} - 2$; $t = 2^{k+2} - 1$) such that $\text{Sq}^{2^k} \text{Sq}^{2^{k-1}} \text{Sq}^1(v_s)$ is an integer class for every $k \geq 5$.

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DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NEW YORK, 10458

Current address: Departamento de Matemáticas, Universidad Autónoma de Santo Domingo, Santo Domingo, Dominican Republic