SOME INTERSECTION PROPERTIES
OF THE FIBRES OF SPRINGER'S RESOLUTION

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Abstract. Combinatorial results are used to calculate the dimension of the intersection of any two irreducible components of the set in the flag variety fixed by the action of a unipotent element of $\text{GL}_n$ whose Jordan decomposition has two blocks. This is then related to the "left cells" of Kazhdan and Lusztig, which are used to construct representations of $S_n$, the Weyl group of $\text{GL}_n$.

0. In this note, we study the fibres of Springer's resolution [St] of the singularities of the unipotent variety in $G = \text{GL}_n(k)$, where $k$ is an algebraically closed field. These fibres are fixed point sets for the action of $G$ on the variety $\mathcal{B}$ parametrizing the complete flags in a vector space of dimension $n$. We use $\mathcal{B}_u$ to fixed by a unipotent element $u \in \text{SL}_n$.

In general, $\mathcal{B}_u$ has several irreducible components. Suppose that the Jordan decomposition of $u$ has block sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, where $\lambda_1 + \cdots + \lambda_s = n$. (We refer to this as the shape $\lambda$, and, by abuse of language, we say that $u$ has shape $\lambda$.) Then each component of $\mathcal{B}_u$ has dimension $\sum_{i=1}^s (i-1)\lambda_i$, and there is one component for each standard Young tableau of shape $\lambda$ (see [S] and below). However, one does not know, in general, the codimension of the intersection of two components; only the "one-hook" case has been done [V]. Here, we calculate the codimension when the Jordan form of $u$ has two blocks. This calculation depends on combinatorial techniques exposed in [LS]. The precise result is (2.1).

This calculation has two-fold significance. First, it enables one, in this case, to verify a conjecture of Kazhdan and Lusztig [KL, 6.3] concerning the configuration of components of $\mathcal{B}_u$; see (4.3). Second, this casts new light on the combinatorial results in [LS] which are used to calculate Kazhdan-Lusztig polynomials, because there is no mention of the geometry of the Grassmannian here.

1.0. In this section, we begin to associate (following [LS]) components of $\mathcal{B}_u$ (for certain $u$) to words in the letters $a$ and $\beta$ (and vice-versa). Let $M_o^b$ denote the set of words made up of $a$ $a$'s and $b$ $\beta$'s, e.g., $M_2^1 = \{ \beta a a, a \beta a, a a \beta \}$. We associate a permutation of $(1, 2, \ldots, a + b)$ to each $w \in M_o^b$ by writing $1, \ldots, a$ in order from right to left under the $a$'s, and then $a + 1, \ldots, a + b$ in order from right to left under the $\beta$'s. Thus, $a a \beta a \beta \alpha a$ is associated to $4, 3, 7, 2, 6, 5, 1$, which can be written $(14237)(56)$.

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(1.1) Suppose that $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ is a shape. A *tableau* of shape $\lambda$ consists of $s$ top-justified columns of respective sizes $\lambda_1, \ldots, \lambda_s$. Thus $(4 \geq 3 \geq 1 \geq 1)$ corresponds to

A *standard Young tableau* of shape $\lambda$ is a tableau filled with the integers $1, \ldots, \Sigma \lambda_i$ such that the rows decrease from left to right and the columns decrease from top to bottom (this is not the usual convention). Thus, the standard Young tableaux of shape $3 \geq 2$ are

(1.2) There is a one-to-one correspondence between pairs of Young tableaux with $n$ boxes and elements of $S_n$, the symmetric group; this is the Robinson-Schensted correspondence (see, e.g. [Kn]). We use the following nonstandard insertion algorithm: let $t_1, \ldots, t_s$ be the top row of a tableau $T$, and let $T'$ be the remainder of $T$. We insert $l$ as follows: if $l > t_1$, the resulting tableau has top row $l, t_2, \ldots, t_s$, and $t_1$ is inserted into $T'$. If $t_j > l > t_{j+1}$, the top row of the resulting tableau is $t_1, \ldots, t_j, l, t_{j+2}, \ldots, t_s$, and $t_{j+1}$ is inserted into $T'$. (It is convenient to assume $-\infty$ is at the end of each row.) To create a pair of tableaux corresponding to $\sigma \in S_n$, insert $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$ in the left tableau while filling the right tableau with $n, n-1, \ldots, 1$ as each new cell is created. To construct a permutation from a tableau pair, exchange the left and right tableaux and imagine how each element was inserted, starting with the right tableau entry whose place is occupied by 1 in the left tableau.

(1.3) Example. We construct the tableau pair for $a \alpha \beta a \beta$. The permutation $\sigma$ has the following action: $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 5, \sigma(4) = 1, \sigma(5) = 4$. We insert 4, 1, 5, 2 and 3 into a tableau, with the result:
Conversely, take the pair

\[
\begin{array}{cccc}
5 & 4 & 5 & 3 \\
3 & 2 & 4 & 2 \\
1 & 1 & & \\
\end{array}
\]

The 1 on the right was "bumped" by 2, which was bumped by 3, so \(\sigma(1) = 3\), etc.

(1.4) **Proposition.** The tableaux associated to \(w \in \mathcal{M}_a^b\) have 2 columns, the second of which has length \(\leq b\).

**Proof.** Notice that the longest increasing subsequence in the permutation associated to \(w\) has length 2. The result now follows from [Kn, 5.1.4, exercise 7]. □

(1.5) Let \(u \in \text{SL}_n\) be a unipotent of shape \(\lambda\). Following [S], we set up a one-to-one correspondence between components of \(\mathcal{B}_u\) and Young tableaux of shape \(\lambda\). Let \(e\) be the \(n \times n\) identity, and write \(\tilde{u} = u - e\). Suppose \(T\) is a tableau of shape \(\lambda\) such that 1 is in row \(p\). Then the generic flag \(F_1 \subset F_2 \subset \cdots \subset F_n\) in the component associated to \(T\) satisfies

\[
F_1 \subset \ker(\tilde{u}) \cap \left[\text{image}(\tilde{u}^{-1})\backslash\text{image}(\tilde{u}^p)\right];
\]

\(F_2, \ldots, F_n\) are determined by following this procedure inductively.

(1.6) It is now clear how \(w \in \mathcal{M}_a^b\) corresponds to a component of \(\mathcal{B}_u\), where \(u\) has shape \(\lambda_1 \geq \lambda_2\) with \(\lambda_1 \geq a, \lambda_2 \leq b\). Assume that the first letter of \(w\) is \(a\) (if not, the second columns of the tableaux associated to \(w\) have length \(< b\), since \(n\) is the last integer inserted); then \(w\) corresponds to a pair of tableaux \((T_L, T_R)\) of shape \(a \geq b\). Take the tableau \(T_R\); it corresponds to a component of \(\mathcal{B}_u\). Conversely, a component of \(\mathcal{B}_u\) corresponds to a tableau \(T_R\), and \((T_L, T_R)\) corresponds to some \(w \in \mathcal{M}_a^b\) (that this is unique follows from the proof of (1.4)).

(1.7) We label each \(w \in \mathcal{M}_a^b\) with the integer \(j\) if the \(j\)th, \((j + 1)\)st letters of \(w\) are \(a\beta\); thus \(a\alpha\beta\alpha\beta\) has the labels 2 and 4.

**Remark.** The labels of \(w\) are clearly the left descent set of \(w\) when \(S_n\) is thought of as a Coxeter group of type \(A_{n-1}\).

**Proposition.** The labels of \(w\) are the parabolic lines contained in the component of \(\mathcal{B}_u\) associated to \(w\).

**Proof.** Recall that a component of \(\mathcal{B}_u\) contains parabolic lines of type \(j\) if one is free to choose \(F_j\) such that \(F_{j-1} \subset F_j \subset F_{j+1}\) when \(F_{j-1}, F_{j+1}\) are fixed in \(F_1 \subset \cdots \subset F_n\). It is clear from (1.5) that this is the case exactly when \(j\) is above or at the same level as \(j + 1\) in the tableau associated to the component.

Suppose \(w\) has the label \(j\), and that \(\sigma\) is the associated permutation. Then \(\sigma(j + 1) > \sigma(k)\) for \(k > j + 1\), since there is a \(\beta\) in the \((j + 1)\)st spot and \(\beta\)'s increase from right to left. Thus, the new cell in the tableau is at the bottom of the first column, since \(\sigma(j + 1)\) "bumps" the element in the upper left-hand corner. However, \(\sigma(j) < \sigma(j + 1)\), since all \(\alpha\) are less than all \(\beta\), so the new cell for \(\sigma(j)\) is at the end of column 2. Since column 2 is always shorter than column 1 (or equal in length), the result follows. □
This can often be used to associate a word to a tableau quickly, since e.g. the tableau

\[
\begin{array}{ccc}
5 & 4 \\
3 & 2 \\
1 \\
\end{array}
\]

indicates lines of type 2 and 4, hence labels 2 and 4, hence $aa\beta a\beta$.

(1.8) PROPPOSITION. Suppose $w \in M^b_a$ begins with $a\beta\beta$. Then $w$ does not correspond to a component of $B_u$ where $u$ has shape $a \geq b$.

PROOF. Suppose the contrary; then the second column of the associated tableau has length $b$. Write $w = a\beta\beta w'$, where $w' \in M^{b-2}_{a-1}$, the second columns of the $w'$ tableaux have length $\leq b - 2$. Insertion of $a + b - 1$, $a + b$, and $a$ into these tableaux give the $w$ tableaux, but the first 2 elements go into the first column, so the second column has length $\leq b - 1$. This is a contradiction. \(\square\)

2.0. A pair $(v, w) \in M^b_a \times M^b_a$ is reduced if all coincident phrases $a\beta$ are removed, e.g. if $(v, w) = (aa\beta a\beta, a\alpha\beta a\beta)$, the reduced pair is $(a\alpha\beta, a\alpha\beta)$. This has the effect of removing common labels.

Each $w \in M^b_a$ is associated to a path in the plane $\mathbb{R}^2$ by starting at $(0,0)$ and assigning segments $(m, n) \rightarrow (m + 1, n - 1)$ to each $a$ and $(m, n) \rightarrow (m + 1, n + 1)$ to each $\beta$, e.g. $aa\beta a\beta$:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

We can now state our result.

(2.1) THEOREM. Associate $v, w \in M^b_a$ to components $X_v, X_w \subset B_u$, where $u$ has shape $a \geq b$.

(a) Suppose the reduced form of $(v, w)$ is

\[
v^{\text{red}} = w_0x_1w_1x_2w_2 \cdots x_mw_m, \quad w^{\text{red}} = w_0\tilde{x}_1w_1\tilde{x}_2w_2 \cdots \tilde{x}_mw_m,
\]

where $x_i \in M^1_a$ and $\tilde{a}\beta = \beta\alpha$, $\tilde{\beta}\alpha = a\beta$. Then

\[
codim_{X_v}(X_v \cap X_w) = m.
\]

(b) If $v^{\text{red}}$ and $w^{\text{red}}$ do not have this form, then the components do not meet.
Remark. The codimension is the number of "diamonds" when the paths of (2.0) are superimposed, e.g.,

\[
\begin{array}{c}
\alpha\beta\alpha\beta \\
\otimes\otimes\otimes\otimes \\
\alpha\alpha\beta\alpha
\end{array}
\]

Superimpose:

Codimension is 2.

3.0.

(3.1) We prove (2.1) by induction on \( b \). When \( b = 1 \), the components have dimension 1, and may be identified by their labels, as each is a parabolic line. Parabolic lines of type \( i \) meet those of type \( j \) when \( j = i + 1 \) or \( i = j + 1 \). The words are

\[
\begin{align*}
\alpha \cdots \alpha\beta\alpha \cdots \alpha & \quad (\beta \text{ in } (i+1)^{\text{st}} \text{ spot}), \\
\alpha \cdots \alpha\alpha\beta\alpha \cdots \alpha & \quad (\beta \text{ in } (i+2)^{\text{nd}} \text{ spot}).
\end{align*}
\]

The conclusion follows.

This also proves (2.1)(b).

(3.2) Notice that the removal of a label reduces the component dimension by 1 (by (1.8), no word begins \( \alpha\beta\beta \)). If components \( X_1, X_2 \) both contain parabolic lines of type \( j \), and if \( \mathcal{F} = (F_1 \subset F_2 \subset \cdots \subset F_{j-1} \subset F_j \subset F_{j+1} \subset \cdots \subset F_n) \) is a flag in \( X_1 \cap X_2 \), then so is \( F_1 \subset \cdots \subset F_{j-1} \subset F_j^c \subset F_{j+1} \subset \cdots \subset F_n \). Thus, the existence of a common label adds 1 to the dimension of each component and 1 to the dimension of the intersection, so the codimension is unchanged. Thus, we can assume that \( v \) and \( w \) are reduced.

(3.3) Suppose that

\[
\begin{align*}
v = v_0x_1v_1 \cdots x_nv_m, \\
w = v_0\bar{x}_1v_1 \cdots \bar{x}_n\bar{v}_m.
\end{align*}
\]

The word \( v_m \) has the form \( \beta \cdots \beta\alpha \cdots \alpha \), since it has no label. Suppose it has \( p \beta \)'s and \( q \alpha \)'s; then by (1.7), the subspaces \( F_{n-p-q} \subset \cdots \subset F_n \) of a general flag in each component are forced, and we can reduce to the case of a unipotent of shape \( a - p \leq b - q \) acting on \( F_{n-p-q} \); the result is known here by induction.

Thus, renaming if necessary, we are reduced to the case

\[
\begin{align*}
v = v_0x_1v_1 \cdots v_{m-1}\beta\alpha, \\
w = v_0\bar{x}_1v_1 \cdots v_{n-1}\beta\alpha.
\end{align*}
\]
The tableau describing $X_v$ is

\[
\begin{array}{cc}
\text{n} & \text{n-1} \\
\vdots & \vdots \\
\end{array}
\]

and that for $X_w$ is

\[
\begin{array}{cc}
\text{n} & \text{n-1} \\
\vdots & \vdots \\
\end{array}
\]

Choose $F = (F_1 \subset \cdots \subset F_n)$ in the intersection. $F_n$ is the space on which $u$ acts, while $F_{n-1}$ is forced by the tableau for $w$. However, there is a $P^1$ of choices for $F_{n-1}$ in the component for $v$, since it contains parabolic lines of type $n - 1$. Thus, we lose 1 dimension of choice in $X_w$, so

$$\text{codim}_{X_v}(X_v \cap X_w) = 1 + \text{codim}_{X_v}(X_v \cap X_w),$$

where

$$\bar{v} = v_0 x_1 v_1 \cdots v_{m-1}, \quad \bar{w} = v_0 \bar{x}_1 v_1 \cdots v_{m-1}.$$ 

This induction step completes the proof. 

4.0. Here is an application of (2.1). Let $l: S_n \to \mathbb{Z}$ be the length function. For each $y, w \in S_n$, there is a Kazhdan-Lusztig polynomial $P_{y,w} \in \mathbb{Z}[q]$ whose degree does not exceed \(\frac{1}{2}(l(w) - l(y) - 1)\) for $y < w$ in the Bruhat order. Let $\mu(y, w)$ be the coefficient of the term of that degree in $P_{y,w}(q)$, and, following [KL, §1], define an “$S_n$-graph” as follows: the nodes are elements of $S_n$, and elements $y$ and $w$ are connected by an edge when either $\mu(y, w)$ or $\mu(w, y)$ is nonzero. Further, each node is labeled with its left descent set.

(4.1) Define another family of graphs as follows: let $u \in SL_n$ be a unipotent. The nodes are components of $\mathcal{B}_u$, and two such are connected if the components meet in codimension 1. The nodes are labelled with the set of parabolic lines contained in the associated component.

(4.2) Suppose that $y_1, y_2 \in S_n$ are permutations constructed, as in (1.0), from $w_1, w_2$.

**Theorem [LS].** Suppose the reduced forms of $w_i$ are

$$w_1 = v_0 x_1 v_1 \cdots x_m v_m, \quad w_2 = v_0 \bar{x}_1 v_1 \cdots \bar{x}_m v_m,$$

with $x_i \in M^1_1$. Then $\mu(y_1, y_2) = 1$ when $m = 1$; otherwise $\mu(y_1, y_2) = 0$.  

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In [KL, 6.3] it is suggested that each graph in (4.1) should be a "W-graph", as in (4.0). It is an immediate consequence of (2.1) and (4.2) that this is the case when \( u \) is a unipotent with shape \( a \geq b \). In particular, the intersection graph of \( \mathcal{B}_u \) will be exactly the "left cell" containing the involution corresponding to \( w = \alpha \cdots \alpha \beta \cdots \beta \), where \( w \) has \( a \) \( \alpha \)'s and \( b \) \( \beta \)'s (this is an involution since it is a product of the longest elements of \( S_a \) acting on \( \{1, \ldots, a\} \) and \( S_b \) acting on \( \{a+1, \ldots, a+b\} \)). See (1.7) to verify that the labelling of the nodes is correct.

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