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ON FREE SUBSEMIGROUPS OF SKEW FIELDS

L. MAKAR-LIMANOV[1]

ABSTRACT. I will prove in this note that the multiplicative group of a skew field with uncountable center either contains a free nonabelian subsemigroup or is commutative.

Let T be a skew field with center Z and let us suppose that Z is an uncountable set. Then the following statement is true.

THEOREM. *If $T \neq Z$ then the multiplicative group of T contains a free subsemigroup with two generators.*

The proof of the theorem is broken into the following three lemmas.

LEMMA 1. *Suppose T has no such subsemigroup. Then for any pair of elements x, y from T there exists a semigroup relation $A(a,b) = B(a,b)$ such that this relation is valid for every pair $x+c,\ y+dx$, where $c, d \in Z$, and that* $\deg_a A = \deg_a B,\ \deg_a A = \deg_b B$.

PROOF. Let us fix a pair x, y and consider all pairs $x+c,\ y+dx$. Each pair fails to span a free semigroup (with two generators) so it satisfies some relation which of course depends on c and d. Now we have only a countable number of possible relations. This means that if we fix, for example, $c = c_0$ then there exists a relation which is satisfied by an infinite number of pairs $x+c_0,\ y+dx$. Let us represent this relation as a polynomial in d with coefficients depending on x, y and c_0. This polynomial has infinitely many roots so all of its coefficients must be zeros (we can use Vandermond's determinant with entries $\{d_i^j\}$). So this relation is satisfied by the pair $x+c_0,\ y+dx$ for every d in Z. Thus for every c there exists a relation $P(c)$ of this type which is satisfied by every pair $x+c,\ y+dx$ (where c is fixed and d can be taken arbitrarily). Let us call such relations *good* for x, y and c. Among the countably many good relations we can find a relation P which is satisfied by $x+c,\ y+dx$ for infinitely many c (and any d). Then the same reasoning shows that this relation P is satisfied by $x+c,\ y+dx$ for every c, d (because for every fixed d this relation is satisfied for infinitely many c and thus for every c). Let us call such relations *better* for x, y. Now for each pair $ex,\ fy$, where $e,\ f \in Z$, there exists a better relation for $ex,\ fy$ and again we can see that there exists a better relation which is satisfied by every pair $ex,\ fy$. By factoring out e and f we can see that this relation has the same degrees in a and in b on its left and right sides.

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LEMMA 2. *If the relation $A(a,b) = B(a,b)$ is nontrivial and satisfies the conclusion of Lemma 1 for a pair $x,\ y \in T$ then there exists some $n = n(x,y)$ such that $(ad_x)^n y = 0$, where $ad_x y = [x,y] = xy - yx$.*

PROOF. We can write $A(a, b + da)$ as $\sum_j A_j(a,b)d^{J-j}$, where $\deg_b A_j = j$. Let us show first that $A_1(a,b)$ uniquely defines $A(a,b)$ if $\deg_b A \neq 0$. Suppose that $A(a,b) = a^{i_1}b^{j_1}\cdots a^{i_k}b^{j_k}$. Then $A_1(a,b) = \sum_{m\in M} a^m ba^{I+J-m-1}$, where $I = \sum i_s,\ J = \sum j_s$. The set M can be represented as the union of the longest possible arithmetic progressions with difference equal to one. The first of these progressions starts with i_1 and contains j_1 terms, the second one starts with $i_1 + j_1 + i_2$ and contains j_2 terms, the mth progression starts with $i_1 + j_1 + \cdots + j_{m-1} + i_m$ and contains j_m terms. This information, together with the known $\sum(i_s + j_s)$, uniquely defines all i's and j's.

Now we may assume that both $\deg_a A$ and $\deg_b A$ are nonzero because otherwise the relation is trivial. So $A_1 = B_1$ is a nontrivial relation. We can rewrite this relation as the equivalent relation

$$(*)\qquad \sum_{s=0}^{I+J} k_s a^s ba^{I+J-s} = 0.$$

(Here all the k_s are 0 or ± 1 but we are going to use only that $k_s \in Z$.) We can rewrite relation $(*)$ as

$$\sum_{s=0}^{I+J-1} k'_s a^s b_1 a^{I+J-1-s} + k'_{I+J} a^{I+J} b = 0,$$

where $b_1 = [b,a]$. Now $k'_{I+J} c^{I+J} y = 0$ which means that $k'_{I+J} = 0$. So instead of the relation $(*)$ we obtain the relation

$$(**)\qquad \sum_s k'_s a^s ba^{I+J-1-s} = 0$$

of degree smaller than the degree of relation $(*)$ and which is satisfied by every pair $a = x + c,\ b = [y,x]$. Now we may apply induction on the degree $\deg_a$ of the relation. The inductive hypothesis is that from every relation of type $(*)$ which is satisfied by all pairs $a = u + c,\ b = v$, where u and v are any fixed elements of T and c is a nonfixed element of Z, it follows that $(ad_u)^n v = 0$ for some n. The base of induction, when this degree is one, is trivial. For the induction step we remark that if we start with a nontrivial relation then we obtain a nontrivial relation again because in $(*)$ and $(**)$ the first nonzero coefficients k_s and k'_s have the same subscript and are equal to each other. This finishes the proof of this lemma.

Now if we assume that T does not contain a free subsemigroup with two generators, we can deduce with the help of Lemmas 1 and 2 that for any pair $x, y \in T$ there exists $n = n(x,y)$ such that $(ad_x)^n y = 0$. If T is not a field, let us take a pair x, y which does not commute. Let us denote $(ad_x)^{n-2}y((ad_x)^{n-1}y)^{-1}$ by w (here $n > 1$ is the minimal power of ad_x which annihilates y). It is clear that $[x,w] = 1$. Let us denote the skew subfield spanned by x and w over Z by D. The proof of the theorem will be finished if we prove the following lemma.

LEMMA 3. *The skew field D contains a free semigroup with two generators.*

PROOF. If D does not contain such a semigroup, then by Lemma 2 for the elements $t = xw$ and w there exists an n such that $(ad_t)^n w = 0$. But $[t, w] = w$ so $(ad_t)^n w = w$ for any n. This contradiction shows that D contains a desired subgroup and so does T.

REMARK 1. It is possible to prove that every skew field which is finite dimensional over its center and does not coincide with its center contains a noncommutative free subgroup. (This result can be extracted from the papers of A. Lichtman [**1** and **2**], see also [**3**].) So the following conjecture sounds quite reasonable: every skew field which is really skew contains a noncommutative free subsemigroup.

REMARK 2. The theorem stated can be looked at as a generalization of a result of I. Kaplansky [**4**] although the restriction on the cardinality of the center somewhat spoils the picture.

REFERENCES

1. A. Lichtman, *Normal subgroups of the multiplicative group of a skew field*, Soviet Math. Dokl. **153** (1963), 1425–1429.
2. ——, *Free subgroups of normal subgroups of the multiplicative group of skew field*, Proc. Amer. Math. Soc. **71** (1978), 174–178.
3. L. Makar-Limanov, *On free subgroups of skew fields*, preprint.
4. I. Kaplansky, *A theorem on division rings*, Canad. J. Math. **3** (1951), 290–292.

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202