

THE CARATHÉODORY DISTANCE DOES NOT DEFINE THE TOPOLOGY

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ABSTRACT. We construct an analytic space X such that the Carathéodory pseudo-distance c_X is a true distance on X ; however, c_X does not define the analytic space topology of X .

1. Introduction. In [2], S. Kobayashi gives a list of problems about intrinsic distances on an analytic space. Some of these problems have been solved by T. Barth [1] and the author [4]. In a similar vein, T. Barth [1] and N. Sibony [3] asked the following question: Let X be a finite-dimensional analytic space for which the Carathéodory distance c_X is a true distance. Does c_X define the analytic space topology of X ? In [3], N. Sibony proved that the answer is affirmative provided X is finitely compact with respect to c_X . In this paper we shall prove that, in general, the answer is negative.

2. The example. Take real numbers $\alpha, \beta, \gamma, \delta$ such that $0 < \beta < \alpha < 1$, $0 < \gamma < \alpha < 1$, and $0 < \delta < 1$. Let D be the union of these three open sets in \mathbf{C}^2 (coordinates z_1, z_2):

$$\begin{aligned} D_1 &= \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1| < 1, |z_2| < \beta\}; \\ D_2 &= \{(z_1, z_2) \in \mathbf{C}^2 \mid 1 > |z_1| > \delta, |z_2| < 1\}; \\ D_3 &= \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1| < 1, \alpha < |z_2| < 1\}. \end{aligned}$$

Similarly, let E be the union of these three open sets in \mathbf{C}^2 (coordinates z_2, z_3):

$$\begin{aligned} E_1 &= \{(z_2, z_3) \in \mathbf{C}^2 \mid 0 < |z_3| < 1, |z_2| < \gamma\}; \\ E_2 &= \{(z_2, z_3) \in \mathbf{C}^2 \mid 1 > |z_3| > \delta, |z_2| < 1\}; \\ E_3 &= \{(z_2, z_3) \in \mathbf{C}^2 \mid |z_3| < 1, \alpha < |z_2| < 1\}. \end{aligned}$$

Let X be the analytic space obtained by patching together D and E as two planes cutting transversely along the z_2 -axis for $\alpha < |z_2| < 1$. It is easy to see that the Carathéodory pseudodistance c_X is a true distance on X . However, by using the extension theorem for holomorphic functions, we can prove the following. Let x_n be the image in X of the point $(0, 1/n)$ in E . Let 0 be the image in X of the origin $(0, 0)$

Received by the editors May 19, 1983.

1980 *Mathematics Subject Classification*. Primary 32H15.

Key words and phrases. Carathéodory distance, topology defined by the Carathéodory distance.

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in D . Then $c_X(x_n, 0) \rightarrow 0$, but x_n does not converge to 0 in the analytic space topology. Thus we have proved the following theorem.

THEOREM. *On X , the Carathéodory pseudodistance c_X is a distance, but c_X does not define the analytic space topology.*

REMARK 1. Unfortunately, I do not know how to construct an example in which X is a complex manifold.

REMARK 2. By [1] this shows that the Carathéodory distance c_X is not inner, answering a question in [2]. Of course, this question is already answered in [1 and 4]. In fact, in [4] I constructed a bounded domain, finitely compact with respect to its Carathéodory distance, such that this distance is not inner.

REFERENCES

1. T. Barth, *Some counterexamples concerning intrinsic distances*, Proc. Amer. Math. Soc. **66** (1977), 49–53.
2. S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.
3. N. Sibony, *Prolongement de fonctions holomorphes et métrique de Carathéodory*, Invent. Math. **29** (1975), 205–230.
4. J. P. Vigué, *La distance de Carathéodory n'est pas intérieure*, Math. Results **6** (1983), 100–104.

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