FINE AND NONTANGENTIAL CONVERGENCE ON
AN NTA DOMAIN

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Abstract. The recent article by Jerison and Kenig on “Boundary behaviour of
harmonic functions in nontangentially accessible domains” did not consider the
relation between fine limits and nontangential limits. The results in this direction
obtained by Hunt & Wheeden [5] for Lipschitz domains are extended here to NTA
domains.

In [6], Jerison and Kenig define a class of bounded domains $D$ (the so-called
bounded NTA domains) for which the topological boundary is the Martin boundary
of $D$ (see [6, Theorem 5.9]). Further, they show that every positive harmonic function
$u$ has a nontangential limit $d\omega$-a.e. where $\omega$ is the harmonic measure of a fixed point
$x_0 \in D$. This limit is identified with $du/d\omega$ if $\mu$ is a positive measure on $\partial D$ that
 corresponds to $u$. Their proof involves a classical maximal function argument.

For Lipschitz domains these results were obtained earlier by Hunt and Wheeden
[5]. They also showed [5, Theorem 5.5] that for $u$ a positive harmonic function, $u$ has
a semifine limit at a boundary point $b$ if and only if $u$ has a nontangential limit at $b$.
In addition, they proved [5, Theorem 5.7] that for any function $u$ on a Lipschitz
domain, if it has nontangential limits at each point of $E \subset \partial D$ then $d\omega$-a.e. it has the
 same fine limits on $E$ (here $\omega$ is the harmonic measure of a fixed point $x_0$). These
results extended earlier work of Brelot and Doob [1] for the half space $\mathbb{R}^n \times \mathbb{R}^+.$

The main purpose of this note is to complement the work of Jerison and Kenig by
establishing these additional results for all NTA domains. The usual method [1] by
which it is shown that the existence of nontangential limits implies the existence of
fine limits a.e. works for NTA domains. As a result, the method used by Brelot and
Doob [1] to prove a local Fatou theorem from the Fatou-Naim-Doob theorem can
be applied to NTA domains (see Theorem 5.1).

It will be assumed without further comment that the value of a constant $C$ (say)
can change from one use to another. If $E$ is a set $E^c$ denotes its complement. For the
definition of potential theoretic concepts the reader is referred to Helms [4].

1. Some properties of NTA domains. A bounded domain $D \subset \mathbb{R}^n$ is said to be
$(M, r_0) =$ nontangentially accessible (or NTA) [6] if there exists $M$ and $r_0$ such that
the following three conditions are satisfied:
(1) (The corkscrew condition). If $r < r_0$ and $q \in \partial D$, then there is a point $a = a(r, q) \in D$ such that (i) $rM^{-1} < |a - q| < rM$ and (ii) $\text{dist}(a, \partial D) > rM^{-1}$.

(2) The complement of $D$ satisfies (1).

(3) (Harnack chain condition). If $x_1, x_2 \in D$ with $\text{dist}(x_i, \partial D) > \varepsilon$ and $|x_1 - x_2| < 2^k \varepsilon$, then there are $Mk$ balls $B_j \subset D$ such that

- $x_1$ is the centre of $B_1$ and $x_2$ is the centre of $B_{Mk}$,
- $B_j \cap B_{j+1} = \emptyset$, $1 \leq j \leq Mk - 1$,
- $M^{-1} \text{diam } B_j \leq \text{dist}(B_j, \partial D) \leq M \text{diam } B_j$ and
- $\text{diam } B_j \geq M^{-1} \text{dist}(x_i, \partial D)$ \quad $(i = 1, 2)$.

If $x \in \mathbb{R}^n$ let $B(x; r) = \{y ||x - y| < r\}$ and let $\Delta(q; r) = B(q; r) \cap \partial D$ if $q \in \partial D$. For $x \in D$ let $\omega(x, A)$ be the harmonic measure of $A \subset \partial D$ associated with $x$, i.e. $\omega(-, A)$ is the solution of the Dirichlet problem with boundary value $1_A$.

Remark 1. The Harnack chain condition has the following immediate consequence. Let $A \subset D$ and assume $\text{dist } A < C \text{dist}(A, \partial D)$. Then, for any positive harmonic function $u$ on $D$ there is a constant $C_2$ such that $u(x) < C_2$ for all $x, y \in A$.

2. If $D$ is an $(M, r_0)$-NTA domain and $T(x) = Xx + a$ with $X > 0$, then it is clear that $T(D)$ is an $(M, \lambda r_0)$-NTA domain.

3. If $M' > M$ and $r_0' < r_0$, an $(M, r_0)$-NTA domain is also an $(M', r_0)$-NTA domain. It will be assumed from now on that $M > 1 \geq r_0$ unless otherwise specified.

It will be useful in what follows to make use of the following consequence of the Boundary Harnack Principle [6, 5.1].

**Proposition 1.1.** Let $D$ be an $(M, r_0)$-NTA domain with $0 \in \partial D$ and not contained in $B(0; 2)$. Let $c < r_0/2(2 + M^2)$ and set

$$A = \{x | \text{dist}(x, \partial D) > c/2, |x| \leq 3/2\}.$$ 

Let $u, v > 0$ be harmonic on $D$ and vanish continuously at $\partial D \setminus B(0; 1/2)$. Let $a \in A$. Then there is a constant $C = C(M, r_0)$ such that

$$1/C \leq u(x)v(a)/u(a)v(x) \leq C \quad \text{for all } x \in D, |x| = 1.$$ 

**Proof** (for the reader's convenience). The corkscrew condition implies $A \neq \emptyset$. For $x \in A$ the Harnack chain condition shows that $u(x)$ is bounded.

If $|x| = 1$ and $x \in A$, then there is a point $q \in \partial D$ with $|x - q| < c$. Since $1 - c > M^2c + 1/2$ (note that $r_0 \leq 1$) it follows that $B(q; M^2c) \subset B(0; 1/2)$. Let $a' = a(cM, q)$ (see the corkscrew condition). Then $\text{dist}(a', \partial D) \geq c$ and $a' \in A$ since $|a' - q| < cM^2$.

It then follows from [6, 4.10] that $u(x)$ is bounded for all $x \in D, |x| = 1$.

**Proposition 1.2.** Let $D$ and $A$ be as in Proposition 1.1. Let $u > 0$ be harmonic on $D$ and vanish continuously at $\partial D \setminus B(0; 1/2)$. Let $a \in A$. Then there is a constant $C = C(M, r_0)$ such that

$$u(x) \leq Cu(a) \quad \text{for all } x \in D, |x| = 1.$$
Proof. The argument used to prove Proposition 1.1 applies. One takes \( a' = a(q, cM^2/2) \) and instead of \([6, 4.10]\) one uses \([6, 4.4]\).

The proof of Lemma 4.1 in [6] shows that there are barriers (see [4] for definition) at any boundary point \( b \) of \( D \) that vanish at \( b \) in a uniform fashion. An immediate consequence of this fact is the next result.

**Proposition 1.3.** Let \( D \) be an \((M, r_0)\)-NTA domain with \( 0 \in \partial D \) and not contained in \( B(0; 2) \). Let \( h \) denote the harmonic function on \( B(0; 1) \cap D \) with boundary values 1 on \( \partial B(0; 1) \cap D \) and 0 elsewhere.

Let \( 0 < c < 1 \). If \( \varepsilon > 0 \), then there exists \( \delta = \delta(\varepsilon, c, M, r_0) \) such that \( h(x) < \varepsilon \) if \( \text{dist}(x; \overline{B}(0; c) \cap \partial D) < \delta \).

**Proof.** Let \( 2r_1 = \min(r_0, 1 - c) \). It follows from [6, 4.1] that \( h(x) \leq M(|x - q| r_1)^\beta \) if \( |x - q| < r \) and \( |q| \leq c, q \in \partial D \).

**Corollary 1.4** (cf. [6, 4.2]). Let \( 1 < c < 2 \) and \( q \in \partial D \). Then, there is a constant \( C_2 \) with \( \omega(x, \Delta(q, c_1r)) \geq C_2 \) if \( x \in \overline{B}(q; r) \cap D \) as long as \( r < \min(1/C_1, r_0) \).

**Proof.** It follows (by scaling) from Proposition 1.3 that \( \omega(x, \Delta(q, c_1r)) \geq 1/2 \) if \( \text{dist}(x; \overline{B}(q; r) \cap \partial D) < c_1r \delta \) where \( \delta = \delta(1/2, 1/C_1, M, r_0) \). The Harnack chain condition, applied to \( \overline{B}(q; r) \cap \{ x \in D \mid \text{dist}(x, \partial D) \geq c_1r \delta \} \), completes the argument.

**Proposition 1.5.** Let \( D \) be an \((M, r_0)\)-NTA domain with \( 0 \in \partial D \) and not contained in \( B(0; 2) \). Let \( k \) denote the harmonic function on \( D \cap \overline{B}(0; 1) \) with boundary values 1 on \( \partial B(0; 1) \cap D \) and 0 elsewhere. Let \( 1 < c < 2 \). If \( \varepsilon > 0 \) then there exists \( \delta = \delta(\varepsilon, c, M, r_0) \) such that \( k(x) < \varepsilon \) if \( \text{dist}(x, \overline{B}(0; c) \cap \partial D) < \delta \).

**Proof.** It is similar to the proof of Proposition 1.3.

### 2. A bubble set is not minimally thin

**Definition 2.1.** Let \( \alpha > 0 \) and \( b \in D \). The set \( \Gamma_\alpha(b) = \{ x \in D \mid |x - b| < (1 + \alpha)\text{dist}(x, \partial D) \} \) is called a nontangential region (or corkscrew) at \( b \) (see [6, 5.6]). Let \( \Gamma_\alpha(b; h) = \Gamma_\alpha(b) \cap B(b; h) \).

A function \( u \) on \( D \) has nontangential limit \( \lambda \) at \( b \) if \( u(x) \to \lambda \) as \( x \to b \) through any corkscrew \( \Gamma_\alpha(b) \) for which \( b \in \Gamma_\alpha(b) \).

**Definition 2.2.** A sequence \( \langle x_n \rangle \) converges to \( b \in \partial D \) nontangentially if there exists a constant \( C > 1 \) and a sequence \( \langle r_n \rangle \subset \mathbb{R}_+ \) converging to zero such that

(i) \( r_nC^{-1} < |x_n - b| < r_nC \),

(ii) \( \text{dist}(x_n, \partial D) > r_nC^{-1} \).

**Lemma 2.3.** A function \( u \) on \( D \) has nontangential limit \( \lambda \) at \( b \in \partial D \) if and only if \( \lambda = \lim_{x \to b} u(x) \) whenever \( \langle x_n \rangle \) converges to \( b \) nontangentially.

**Definition 2.4.** A bubble set at \( b \in \partial D \) is a set \( B = \bigcup_{n=1}^{\infty} B_n \), where \( B_n = B(x_n; \gamma r_nC^{-1}) \) with \( 0 < \gamma < 1 \) and \( \langle x_n \rangle \) a sequence converging to \( b \) nontangentially.

**Notation.** If \( u \geq 0 \) is superharmonic on \( D \), let \( R_u \) denote \( \inf(v(x) \mid v \geq 0, \text{superharmonic on } D, v \geq u \text{ on } E) \). (It is called the réduite of \( u \) relative to \( E \) and is denoted by \( R^+_u \) in [4].)
Lemma 2.5. Let \( \langle x_n \rangle \) converge to \( b \in \partial D \) non-tangentially and let \( B_n = B(x_n; \gamma r_n C^{-1}) \) be the \( n \)-th ball in an associated bubble set.

Assume \( B' \cap B_n \neq \emptyset \) where \( B' \) is a ball \( \subset D \) and \( M^{-1} \text{diam} B' \leq \text{dist}(B', \partial D) \leq M \text{diam} B' \). Then there is a constant \( N = N(\gamma, C, M) > 0 \) such that

\[
R_{B_n}(x) \geq N \quad \text{if} \quad x \in B'.
\]

Proof. Let \( r' \) be the radius of \( B' \). Then \( M^{-1}2r' \leq \text{dist}(B', \partial D) \leq |x_n - b| + \gamma r_n/C \leq r_n(C + \gamma C^{-1}) \). Translate the centre of \( B' \) to the origin and scale by \( 1/r' \). The image \( B'_n \) of \( B_n \) is a ball of radius \( \gamma r_n C^{-1}/r' \geq R > 0 \), where

\[
R = 2\gamma M^{-1}C^{-1}(C + \gamma C^{-1})^{-1} = R(\gamma, C, M).
\]

Hence, to prove the lemma it suffices to solve the following problem. Let \( W = B(0; 1 + \eta) \) and let \( B_t = B(te_t; R) \) with \( 0 < t < R + 1 \). Is there a constant \( N = N(\eta, R) \) independent of \( t \), such that if \( u \) is the equilibrium or capacitory potential (see [4] for definition) on \( W \) of \( B_t \cap W \) then \( u(x) \geq N \) for all \( x, |x| \leq 1 \)? It will be assumed that \( \eta \leq \min(M^{-1}, 1/3R) \). Let \( q_t \) be the equilibrium potential on \( W \) of \( B(te_t; \eta/2) \). For any fixed value of \( t \) it has a minimum on \( \overline{B}(0; 1) \). It suffices therefore to show that there is a finite number of values of \( t \), say, \( t_1, \ldots, t_m \) where \( m = m(n, R) \) such that \( |x| < 1 \Rightarrow u(x) \geq q_{t_i}(x) \) for some \( i = 1, \ldots, m \). To see this let \( t_1 = 1 + \eta/2, t_{k+1} = t_k - 2R + \eta \).

Corollary 2.6. Let \( D \) be an \((M, r_0)\)-NTA domain. Let \( \langle x_n \rangle \) converge to \( b \in \partial D \) non-tangentially and let \( B_n = B(x_n; \gamma r_n C^{-1}) \) be the \( n \)-th ball in an associated bubble set.

Let \( r < r_0 \) and let \( 2r_n C < r \) for \( n \geq n(r) \). If \( u > 0 \) is harmonic on \( D \) and \( w = R_{B_n} \), then there is a constant \( C_1 = C_1(M, r_0, \gamma, C) \) such that

\[
(*) \quad C_1^{-1} \leq \frac{u(x)w(x_n)}{u(x_n)w(x)} \leq C_1
\]

for all \( x \in D, |x| = 2r_n(C + \gamma C^{-1}) \) providing \( u \) vanishes continuously at \( \partial D \setminus \{b\} \).

Proof. Let \( C \) be the constant in Definition 2.2 associated with \( \langle x_n \rangle \). Let \( \lambda^{-1} = 2r_n(C + \gamma C^{-1}) \). Set \( T(x) = \lambda(x - b) \) and consider the image of \( D \) under \( T \). If the constant \( c \) of Proposition 1.1 is taken less than \( 1/(2C^2 + 2\gamma) \), then \( T(x_n) \in \mathcal{A} = \{x | \text{dist}(x, \partial T(D)) \geq c/2, |x| \leq 3/2 \} \).

The argument of Proposition 1.1 applies once certain observations have been made. Let \( x_1, x_2 \in D \) and let \( (B'_i) \) be a Harnack chain from \( x_1 \) to \( x_2 \). Then there is a constant \( C \) such that \( u(x_1) \leq Cu(x_2), w(x_1) \leq Cw(x_2) \) where \( u \geq 0 \) is harmonic on \( D \). To see this let \( B'_i \) be the first ball to meet \( B_n \) and \( B'_i \) be the last ball to meet \( B'_n \). For all positive harmonic functions \( v \) on \( D \cap B_n \), there are constants \( C_1, C_2 \) such that \( v(x_1) \leq C_1v(x_2) \leq C_1^2v(x_1) \) for all \( x \in B'_{i-1} \) and \( v(x_2) \leq C_2v(y) \leq C_2^2v(x_2) \) for all \( y \in B'_{i+1} \). Replacing \( B'_{i+1}, \ldots, B'_{i-1} \) by \( B_n \) shows that

\[
w(x_1) \leq C_1w(x) \leq C_1 \leq C_1N^{-1}w(y) \leq C_2C_1N^{-1}w(x_2).
\]

Consequently, the growth of \( w \) is controlled by Harnack chains in the same fashion as the growth of any positive harmonic function.
Jerison and Kenig proved that the Martin Boundary of $D$ (see [4] for definition) can be identified with the Euclidean boundary [6, Theorem 5.9]. Furthermore, all boundary points are minimal (see [4] for definition) in view of [6, Theorem 5.10]. It states that to each point $b \in \partial D$ there corresponds a positive harmonic function $K_b(x)$, normalized by requiring $K_b(x_0) = 1$ for a fixed point $x_0 \in D$. This "Poisson" kernel represents all positive harmonic functions $u$ on $D$. More precisely, to each positive harmonic function $u$ on $D$ there is a unique positive Borel measure $\mu$ with $u(x) = \int K_b(x) \mu(db)$ for all $x \in D$. Consequently, $K_b$ is a minimal harmonic function (i.e., if $0 < u \leq K_b$ and $\Delta u = 0$, then $u = \lambda K_b$ for some $\lambda > 0$).

**Definition 2.7.** A set $E \subset D$ is (minimally) thin at $b \in \partial D$ if there is a super-harmonic function $v$ on $D$ distinct from $K_b$ such that

(i) $v = K_b$ on $E$ and

(ii) $v \leq K_b$ i.e. if $R_F K_b \neq K_b$.

It is well known that the union of two sets that are thin at $b$ is also thin at $b$, since a set $E$ is thin at $b$ if and only if $K_b$ is dominated on $E$ by a potential (cf. [10]). Hence, the sets whose complements are thin at $b$ form a filter $F(b)$.

**Definition 2.8.** A function $\mu$ on $D$ has a fine limit at $b$ if it has a limit along the filter $F(b)$.

The theorem of Fatou-Naim-Doob [2, 9, 10] implies that if $u, h$ are any two positive harmonic functions on $D$ then $\nu$-a.e. $u/h$ has fine limit at $b \in \partial D$ equal to $(d\mu/d\nu)(b)$ where $\mu$ and $\nu$ are the measures such that $u(x) = \int K_b(x) \mu(db)$ and $h(x) = \int K_b(x) \nu(db)$.

**Lemma 2.9 (Brelot - Doob).** Let $b \in \partial D$ and assume that any bubble set at $b$ is not thin at $b$. Let $u, h$ be positive harmonic functions on $D$. Then if $\lambda$ is the fine limit at $b$ of $u/h$ it is the nontangential limit of $u/h$ at $b$.

**Proof (cf. [1, Théorème 3]).** Let $\eta$ be a nontangential cluster value of $u/h$ at $b$. Then there is a sequence $(x_n)$ converging nontangentially to $b$ such that $\eta = \lim_{n \to \infty} u(x_n)/h(x_n)$.

Let $\varepsilon > 0$. By Harnack's inequality there is a bubble set $B$ associated with $(x_n)$ such that $|u(x)/h(x) - \eta| < \varepsilon$ for all $x \in B_n$ and $n$ sufficiently large. Since the bubble set is not thin at $b$, it meets every set in $F(b)$ and so $\lambda = \eta$.

Consequently, in order to deduce the existence of nontangential limits from the Theorem of Fatou-Naim-Doob it suffices to verify the following result.

**Proposition 2.10.** Let $B = \bigcup_{n=1}^{\infty} B_n$ be a bubble set at $b \in \partial D$. Then $B$ is not thin at $b$.

**Proof.** If $B$ is thin at $b$, then there is a potential $p$ on $D$ with $p \geq K_b$ on $B$. Let $u_n = R_p K_b$. The existence of $p$ is impossible if there is a constant $C > 0$ with $u_n(x_0) \geq C$, where $x_0$ is the normalising point for which $K_b(x_0) = 1$.

In view of Harnack's inequality applied to $K_b$ on $B_n$ it follows that there is a constant $C$ with $C^{-1} \leq u_n(x)/u_n(x_0) \leq C$ where $x_n$ is the centre of $B_n$ and $x \in B_n$. It follows from Corollary 2.6 that $u_n(x) \geq CK_b(x)$ for all $x \in D$, $|x| = 2r_n(C + \gamma C^{-1})$. The maximum principle implies that $u_n(x_0) \geq C > 0$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Corollary 2.11. Let $u, h > 0$ be harmonic on $D$ and “Poisson” integrals of the measures $\mu, v$ (respectively). Then $u/h$ has nontangential limit $(d\mu/dv)(b)$ $v$-a.e. In particular, $u$ has nontangential limit $(d\mu/d\omega)(b)$ $\omega$-a.e. where $w = \omega(x_0, -)$ is the harmonic measure corresponding to $x_0$.

Proof. The first statement is an immediate consequence of Proposition 2.10, Lemma 2.9 and the Theorem of Fatou-Naim-Doob.

The second statement is a particular case of the first once it is established that the constant function 1 is represented by $\omega$. Let $\lambda$ be the measure such that $\int K_b(x)\lambda(db) = 1, \forall x \in D$.

The kernel function approach to the determination of the minimal functions $K_b$ makes it obvious that $\lambda = \omega$. However, this fact is always true and in this particular case is easily established from general considerations.

Let $f \in C(\partial D)$, set $h(x) = \int K_b(x)f(b)\lambda(db)$ and $u(x) = \int f(b)\omega(x, db)$, where $\omega(x, db)$ is the harmonic measure corresponding to $x$. Since an NTA-domain is regular (a consequence of [6, 4.1]) (see [4] for definition) the ordinary boundary limit (and hence the fine limit) of $u$ at $\partial D$ is $f$. Consequently, $h - u = v$ is a bounded harmonic function with fine limit equal to zero $\lambda$-a.e. Since every positive harmonic is the (Poisson) integral of a unique positive measure on $\partial D$ this implies $v = 0$. Consequently, $K_b(x)\lambda(db) = \omega(x, db)$ and, in particular, $\lambda = \omega(x_0, -) = \omega$.

3. Semifine convergence and nontangential convergence. Fix $b \in \partial D$ and let $E \subset D$. Set $E_k = \{x \in E|2^{-k-1} \leq |x - b| \leq 2^{-k}\}$.

Definition 3.1. $E$ is said to be semithin at $b$ if $\lim_{k \to \infty} R_{E_k}K_b = 0$.

Since $R_{A_1 \cup A_2}u \leq R_{A_1}u + R_{A_2}u$ for any two sets $A_i \subset D$ and $u \geq 0$ superharmonic on $D$, it follows that the union of two sets that are semithin at $b$ is also semithin at $b$. Let $S(b) = \{E \subset E \text{ semithin at } b\}$.

Definition 3.2. A function $f$ has semifine limit at $b$ if it converges along $S(b)$.

Proposition 3.3. A bubble set at $b$ is not semithin at $b$.

Proof. Let $B = \bigcup_{n=1}^{\infty} B_n, B_n = B(x_n, r_nC^{-1})$ where $(r_n)$ and $C$ are as in Definition 2.2. Then, each $B_n$ meets at most $m + 1$ of the sets $(x \in D|2^{-k-1} \leq |x - b| \leq 2^{-k})$ if $m \log 2 \geq [4 \log C - \log(1 - r)]$. Since $R_{A_1 \cup A_2}u \leq R_{A_1}u + R_{A_2}u$ the result follows from the fact established in the proof of Proposition 2.8: $R_{B_n}K_b(x_0) \geq C > 0$.

Proposition 3.4. Let $E \subset D$ be such that $\operatorname{dist}(x, \partial D)|x - b|^{-1} \to 0$ as $|x - b| \to 0$ (i.e. let $E$ be a tangential set). Then $E$ is semithin at $b$ (the point used to define the sets $E_k$).

Proof. $E$ is a tangential set at $b$ if and only if $2^k \operatorname{dist}(E_k, \partial D) \to 0$ as $k \to \infty$. Let $\delta > 0$ and let $k(\delta)$ be such that $k > k(\delta)$ implies $2^k \operatorname{dist}(x, \partial D) < \delta$ for all $x \in E_k$.

In view of Propositions 1.3 and 1.5 it is possible to choose $\delta$ so that the harmonic function $h_k$ on $D \cap \{x|2^{-k-2} < |x - b| < 2^{-k+1}\}$ with boundary values 1 on $\{B(b; 2^{-k-2}) \cap D\} \cup \{B(b; 2^{-k+1}) \cap D\} \cup 0$ elsewhere is less than $\varepsilon$ on $E_k$.

Let $a_k = a(2^{-k}, b)$. Then by Proposition 1.2 there is a constant $C$ with $K_b(x) \leq CK_b(a)$ if $|x| = 2^{-k-2}$ or $|x| = 2^{-k+1}$. Hence, $x \in E_k$ implies $K_b(x) \leq \varepsilon CK_b(a)$. 

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Let $w_k(x) = w(x, \Delta(b, 2^{-k+1}))$. It follows from Corollary 1.4 that there is a constant with $Cw_k(x) \geq R_{D_1}(x)$ if $|x - b| > 2^{-k+2}$, $D_k = D \cap \{x | 2^{-k-1} \leq |x| \leq 2^{-k}\}$. Consequently,

$$R_{E_k}K_b(x) \leq \varepsilon C K_b(a_k)w_k(x) \text{ for } |x - b| > 2^{-k+2}.$$ 

In particular, $R_{E_k}K_b(x_0) \leq \varepsilon C K_b(a_k)w_k(x_0)$.

Let $a_{k-2} = a(2^{-k+2}, b)$. It follows from [6, 4.2] (or Corollary 1.4) that $w_k(a_{k-2}) \geq C$. Also, by the Harnack chain condition, for some constant $C, K_b(a_k) \leq c K_b(a_{k-2}) \leq C K_b(a_k)$. Scaling by $1/r_k$, $r_k = |a_{k-2}|$, it follows from Proposition 1.1 that $|x - b| > 2^{-k+2}$ implies $w_k(x)K_b(a_k) \leq CK_b(x)$. Since $K_b(x_0) = 1$, the previous estimate for $R_{E_k}K_b(x_0)$ shows that $R_{E_k}K_b(x_0) \leq Ce$.

**Proposition 3.4.** Let $u, h > 0$ harmonic. Then $u/h$ has a semifine limit $\lambda$ at $b$ if and only if $u/h$ has a nontangential limit $\lambda$ at $b$.

**Proof (=>).** This follows from Proposition 3.3 by the argument used to prove Lemma 2.9.

(<=). Let $\lambda$ be a semifine limit value of $u/h = f$ at $b$ i.e. for every $\varepsilon > 0$, $E = \{x | \lambda - \varepsilon < f(x) < \lambda + \varepsilon\}$ meets every set in $S(b)$. It must meet every $\Gamma_a(b)$ for which $b \in \Gamma_a(b)$. Otherwise, for any such $a$, there exists $r(a)$ with $\Gamma_a(b) \cap \{x | \lambda - \varepsilon < f(x) < \lambda + \varepsilon\} \cap B(b, r) = \emptyset$ if $r < r(a)$. In this case, $2^k \text{dist}(E_k, \partial D) \to 0$ as $k \to \infty$. Hence, $\lambda$ is a nontangential limit value of $f$ at $b$.

4. **Nontangential convergence implies fine convergence $d\omega$-a.e.** Let $D$ be a bounded $(M, r_0)$-NTA domain and let $F \subset \partial D$ be closed. Let $W = \bigcup_{b \in F} \Gamma_a(b, h)$.

**Proposition 4.1.** $W$ is thin at $d\omega$-a.e. point of $F$.

**Proof (cf. [1, Théorème 8]).** Assume $(1 + \alpha) \text{dist}(x, \partial D) < h$. If $x \notin W$ then $x \not\in \Gamma_a(b)$ for all $b \in F$. Hence, $|x - b| \geq (1 + \alpha) \text{dist}(x, \partial D)$. Let $q \in \partial D$ be such that $|x - q| = \text{dist}(x, \partial D)$. Then, if $b \in F$,

$$|q - b| + |x - q| \geq |x - b| \geq (1 + \alpha)|x - q|$$ 

and so $|q - b| \geq \alpha|x - q|$. Let $r = \text{dist}(q, F)$. Then, $r \geq \alpha|x - q|$. It follows from [6, 4.2] that $\omega(x, \Delta(q, r/2)) \geq C$ since $x = a(|x - q|, q)$ where $0 < C < 1$.

Let $u$ be the harmonic function with boundary value $1_F$. It follows from the above that if $(1 + \alpha) \text{dist}(x, \partial D) < h$ and $u(x) > 1 - C$ then $x \in W$. The theorem of Fatou-Naim-Doob implies that $\{x | u(x) > 1 - C\}$ is a fine neighbourhood of $d\omega$-a.e. point of $F$. Since $\{x | (1 + \alpha) \text{dist}(x, \partial D) < h\}$ is a neighbourhood (hence a fine neighbourhood) of $\partial D$ the result follows.

**Theorem 4.2.** Let $D$ be an NTA-domain and $f: D \to \mathbb{R}$ a function that has a nontangential limit $\varphi$ on $E \subset \partial D$. Then $d\omega$-a.e. on $E$, $f$ has fine limit $\varphi$.

**Proof.** The argument used to prove Theorem 5.7 in [5] applies.

5. **A local Fatou theorem.** Let $D$ be an NTA-domain and let $W = \bigcup_{b \in E} \Gamma_a(b, h)$ where $h, \alpha$ vary with $b$. Assume $u$ is harmonic on $W$ and lower bounded on each $\Gamma_a(b, h)$. Extend $u$ trivially to $D$ by setting $u(x) = 0$ if $x \in D \setminus W$.  


Theorem 5.1 (cf. [6, Theorem 6.4]). At $d\omega$-a.e. $b \in E$ the extended function $u$ has a nontangential limit as a function on $D$.

Proof. As usual it suffices to consider the case of $E$ closed, $\alpha$ and $h$ independent of $b \in E$ and $u > 0$. Let $E_0 \subset E$ be the set of points $b$ for which $W$ is a fine neighbourhood. $E_0$ can be identified with a subset of the Martin boundary $\partial D$ of $D$ and of $W$ [3, 7] and the null subsets of $F_0$ for $d\omega$ and the measure representing 1 on $W$ coincide. Hence, the theorem of Fatou-Naïm-Doob applied to $u$ on $W$ implies that $d\omega$-a.e. $u$ has a fine limit on $D$ at the points of $E$.

The Calderón Density Lemma (i.e. [6, 6.1]) and the proof of Lemma 2.9 imply that $u$ has nontangential limits a.e. on $E$ (note that in Lemma 2.9 $u$ can be taken to be defined on a corkscrew with large $\alpha$).

Remark. This proof of a local Fatou theorem avoids the problem of constructing sub-NTA domains of $D$ in neighbourhoods of boundary points of $D$ [6, Theorem 3.11] which was solved by Jones [7].

Bibliography