IMAGE AREAS AND $H_2$ NORMS
OF ANALYTIC FUNCTIONS

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Abstract. For an analytic function $f$ in the unit disc $U$ with $f(0) = 0$, the inequality $\|f\|_2^2 \leq \frac{1}{\pi} \text{area}\{f(U)\}$ is shown, where an equality occurs if and only if $f$ is a constant multiple of an inner function. As a corollary, it is shown that for an analytic function in a general domain the square of its $H_2$ norm is bounded by its Dirichlet integral, with the equality condition being settled.

1. Introduction. Let $U$ denote the unit disc and $T$ the unit circle in the complex plane $\mathbb{C}$. It is shown by Alexander, Taylor and Ullman [1, Theorem 1, p. 335] that if $f$ is an analytic function in $U$ with $f(0) = 0$, then

$$\|f\|_2^2 \leq \frac{1}{\pi} \text{area}\{f(U)\}$$

holds, where $\|f\|_2$ denotes the $H_2$ norm

$$\|f\|_2 = \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\}^{1/2}$$

of $f$. An inner function is a bounded analytic function $\psi$ in $U$ such that the Fatou’s boundary function $\psi^*$ satisfies $|\psi^*(e^{i\theta})| = 1$ almost everywhere on $T$. In this paper we shall offer another proof of (1.1) and show that an equality occurs in (1.1) if and only if $f$ is of a form $f = c\psi$, where $c$ is a constant and $\psi$ is an inner function with $\psi(0) = 0$. Our proof depends on Littlewood’s subordination principle and Green’s formula.

Let $D$ be a plane domain with $0 \in D$, which possesses a Green’s function. Following Rudin [5], we denote by $H_2(D)$ the class of functions $F$ analytic in $D$ for which $|F(z)|^2$ admits a harmonic majorant in $D$. For $F \in H_2(D)$, let $h(z)$ be the least harmonic majorant of $|F(z)|^2$ in $D$ and define

$$\|F\|_2 = h(0)^{1/2}.$$ 

It is well known that (1.3) reduces to (1.2) when $D = U$.

In §2 we state our main results. In §3 we state preliminary lemmas, which we use in §4 for the proof of the Main Theorem. In §5, as a corollary to the Main Theorem, the inequality

$$\|F\|_2^2 \leq \frac{1}{\pi} \int_D \int_D |F'(z)|^2 dx dy$$

is shown for $F \in H_2(D)$ with $F(0) = 0$, and its equality condition is settled.
2. Main results. Now we state our main results in the form of a theorem.

**Main Theorem.** If \( f \in H_2(U) \) with \( f(0) = 0 \), then

\[
\|f\|^2 \leq \frac{1}{\pi} \text{area}\{f(U)\},
\]

and an equality occurs if and only if \( f \) is of the form \( f = c\psi \), where \( c \) is a constant and \( \psi \) is an inner function with \( \psi(0) = 0 \).

**Corollary.** If \( F \in H_2(D) \) with \( F(0) = 0 \), then

\[
\|F\|^2 \leq \frac{1}{\pi} \int_D |F'(z)|^2 \, dx \, dy,
\]

where an equality occurs if and only if \( D \) is a domain which is obtained from a simply-connected domain \( W \) by deleting a set of capacity zero and \( F \) is (extended to) a conformal map of \( W \) onto a disc with center at 0.

3. Preliminary results. In this section we state several preliminary lemmas which we use in the next section for the proof of the Main Theorem.

**Lemma 1.** If \( \phi \) is a bounded analytic function in \( U \) with \( |\phi(z)| \leq 1 \), \( z \in U \), and \( \phi(0) = 0 \), then for any \( f \in H_2(U) \)

\[
\|f \circ \phi\|_2 \leq \|f\|_2,
\]

where an equality occurs for some nonconstant \( f \in H_2(U) \) if and only if \( \phi \) is an inner function.

**Proof.** The inequality (3.1) is an easy consequence of Littlewood’s subordination principle (see, say, [2, p. 10]). For the equality condition, see [6, Theorem 3, p. 351 or 4, Theorem 1, p. 316].

Let \( p : U \to D \) be a universal covering map of \( D \) such that \( p(0) = 0 \). The following lemma is well known, for the proof see [5, p. 50 or 4, Lemma 1, p. 316].

**Lemma 2.** If \( F \in H_2(D) \) with \( F(0) = 0 \), then

\[
\|F\|_2 = \|F \circ p\|_2.
\]

The next lemma is an essential part of our proof of the Main Theorem.

**Lemma 3.** If \( I(z) \equiv z \), \( z \in D \), then

\[
\|I\|^2 \leq \frac{1}{\pi} \text{area}(D),
\]

and an equality occurs if and only if \( D \) is a domain of a form \( D = \{|z| < r\} - E \), where \( r > 0 \) and \( E \) is a closed set of capacity zero.

**Proof.** Let \( S \) be a plane domain with smooth boundary \( \Gamma \), and \( u \) and \( v \) be \( C^2 \) functions on the closure \( \overline{S} \) of \( S \). Then Green’s theorem states that

\[
\int_S (v\Delta u - u\Delta v) \, dx \, dy = \int_{\Gamma} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, ds,
\]

where \( \Delta \) denotes the Laplacian, \( \partial/\partial n \) differentiation in the inner normal direction and \( ds \) arc length on \( \Gamma \).
Let $h(z)$ denote the least harmonic majorant of $|I(z)|^2 \equiv |z|^2$ in $D$. Let $G(z)$ be a Green’s function of $D$ with logarithmic singularity at 0, and $G^*(z)$ be its harmonic conjugate, which is locally defined up to an additive constant. Put $P(z) = G(z) + iG^*(z)$, then it is known that $P(z)$ is multiple-valued but it has a single-valued derivative $P'(z)$, which is analytic in $D$, except for a simple pole at 0.

First we assume that $D$ is a regular domain, where “regular” means that the boundary $B$ of $D$ consists of a finite number of mutually disjoint analytic Jordan curves. Let $k$ be a positive integer. We apply Green’s formula (3.4) in $D$ with $u(z) = |z|^2$ and $v(z) = 1 - e^{-2kG(z)}$. Simple calculations show that $\Delta u = 4$ and $\Delta v = -4k^2|P'(z)|^2e^{-2kG(z)}$ in $D$ and that $v = 0$ and $\partial v/\partial n = 2k\partial G(z)/\partial n$ on $B$, which, substituted into (3.4), yields

\[
\int \int_D (1 - e^{-2kG(z)}) \, dx \, dy + k^2 \int \int_D |z|^2 |P'(z)|^2e^{-2kG(z)} \, dx \, dy
\]

(3.5)

\[
= k^2 \int_B |z|^2 \frac{\partial G(z)}{\partial n} \, ds = \pi kh(0).
\]

Put $g(z) = ze^{P(z)}$, then it is multiple-valued but $|g(z)|^2$ is single-valued and subharmonic in $D$, since $\log |g(z)|^2 = 2 \log |z| + 2G(z)$ is harmonic. We again apply Green’s formula (3.4) in $D$ with $u(z) = |g(z)|^2$ and $v(z) = 1 - e^{-2(k+1)G(z)}$. Similarly simple calculations show that $\Delta u = 4|g'(z)|^2$ and $\Delta v = -4(k+1)^2|P'(z)|^2e^{-2(k+1)G(z)}$ in $D$ and that $v = 0$ and $\partial v/\partial n = 2(k+1)\partial G(z)/\partial n$ on $B$, which, substituted into (3.4), yields

(3.6)

\[
\int \int_G |g'(z)|^2 (1 - e^{-2(k+1)G(z)}) \, dx \, dy + (k+1)^2 \int \int_D |z|^2 |P'(z)|^2e^{-2kG(z)} \, dx \, dy
\]

\[
= (k+1)^2 \int_B |z|^2 \frac{\partial G(z)}{\partial n} \, ds = \pi(k+1)h(0),
\]

since $|g(z)|^2 = |z|^2e^{2G(z)}$ for $z \in D$ and $|g(z)|^2 = |z|^2$ for $z \in B$. Here note that the two integrals of the second terms of (3.5) and (3.6) are identical. Therefore, combining the two equalities, we see

\[
\int \int_D (1 - e^{-2kG(z)}) \, dx \, dy
\]

\[
= \frac{\pi k}{k+1}h(0) + \left( \frac{k}{k+1} \right)^2 \int \int_D |g'(z)|^2 (1 - e^{-2(k+1)G(z)}) \, dx \, dy.
\]

On letting $k \to \infty$, Lebesgue’s monotone convergence theorem yields

(3.7)

\[
\frac{1}{\pi} \text{area}(D) = h(0) + \frac{1}{\pi} \int \int_D |g'(z)|^2 \, dx \, dy.
\]

In order to deal with the case of a general domain $D$, let $\{D_n\}$ be a regular exhaustion of $D$ such that $0 \in D_n$, $n = 1, 2, \ldots$. We denote by $G_n(z)$ and $g_n(z)$, respectively, the functions for $D_n$ which correspond to $G(z)$ and $g(z)$ for $D$. Let $h_n(z)$ be the least harmonic majorant of $|I(z)|^2 \equiv |z|^2$ in $D_n$. Then (3.7) for $D_n$ is

\[
\frac{1}{\pi} \text{area}(D_n) = h_n(0) + \frac{1}{\pi} \int \int_{D_n} |g_n'(z)|^2 \, dx \, dy,
\]
in which, letting \( n \to \infty \), we see by Lebesgue’s monotone convergence theorem and Fatou’s lemma

\[
\frac{1}{\pi} \text{area}(D) \geq h(0) + \frac{1}{\pi} \int_D |g'(z)|^2 \, dx \, dy,
\]

since \( h_n(z) \to h(z) \) and \( |g_n'(z)| \to |g'(z)| \) for any \( z \in D \) as \( n \to \infty \). Noting (1.3), we see that (3.8) implies (3.3).

As for the equality condition, the if part is almost trivial. In fact, if \( D \) is as stated in the lemma, then we easily see that \( h(z) \equiv r^2 \), and hence

\[
\|I\|_2^2 = r^2.
\]

On the other hand, \( \text{area}(D) = \pi r^2 \), since a set of capacity zero is of area zero. Thus, an equality occurs in (3.3). Next, assume that an equality holds in (3.3), then we see by (3.8) that \( |g'(z)| \equiv 0 \) in \( D \), and hence \( G(z) = \log(c/|z|) \) for some positive constant \( c \), which means that \( D \) is as stated in the lemma. This completes the proof of the lemma.

4. Proof of Main Theorem. Let \( D = f(U) \) and \( p: U \to D \) be a universal covering map of \( D \) such that \( p(0) = 0 \), as before. By the monodromy theorem, we can determine a single-valued branch of \( p^{-1} \circ f \), which we denote by \( \phi \). Then we easily see that \( \phi \) is a bounded analytic function in \( U \) with \( |\phi(z)| \leq 1 \), \( z \in U \), for which \( f = p \circ \phi \), and hence by Lemma 1

\[
\|f\|_2 \leq \|p\|_2.
\]

Applying Lemma 2 with \( f = I \), we see

\[
\|I\|_2 = \|I \circ p\|_2 = \|p\|_2,
\]

which, combined with (4.1) and Lemma 3, yields (1.1), as asserted.

As for the equality condition, the if part is again almost trivial. In fact, suppose that \( f \) is of the form as stated in the theorem. Since any inner function covers \( U \) with the exception of a set of capacity zero by a theorem of Frostman [3], we easily see

\[
\frac{1}{\pi} \text{area}\{f(U)\} = |c|^2 = \|f\|_2^2.
\]

Next assume that an equality holds in (1.1); then equalities must hold both in (4.1) and (3.3). Therefore, by the equality conditions of Lemmas 1 and 3, we see that \( \phi \) must be an inner function and that \( f(U) \) must be a domain of the form as stated in Lemma 3. Thus, we see that \( |p(z)| \leq r \) for \( z \in U \) and that \( \|p\|_2 = \|T\|_2 = r \) by (4.2) and (3.9), and hence that \( p \) must be expressed as \( p = r \psi \), where \( \psi \) is an inner function. Since the composite function of two inner functions is again an inner function by Lemma 2 (cf. [6, p. 351]), we see that \( f \) must be the form as stated in the theorem.

5. Proof of Corollary. Put \( f = F \circ p \); then by Lemma 2

\[
\|f\|_2 = \|F\|_2.
\]

It is obvious that

\[
\text{area}\{F(D)\} \leq \int_D |F'(z)|^2 \, dx \, dy
\]
and that an equality occurs in (5.2) if and only if $F$ is univalent in $D$. Since $F(D) = f(U)$, we obtain (2.1) by combining (5.1), (5.2) and the theorem. The equality condition immediately follows those of (5.2) and the theorem.

**REMARK.** With a slight modification of the above argument, we can also prove a version of the corollary for the case of Riemann surfaces.

**References**


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