A FUNDAMENTAL INEQUALITY IN THE CONVOLUTION
OF $L_2$ FUNCTIONS ON THE HALF LINE

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ABSTRACT. For any positive integer $q$ and $F_j \in L_2(0, \infty)$, we note the inequality for the iterated convolution $\prod_{j=1}^{2q} F_j$ of $F_j$:

$$\int_0^{\infty} \left| \prod_{j=1}^{2q} F_j(t) \right|^2 t^{1-2q} dt \leq \frac{1}{(2q-1)!} \prod_{j=1}^{2q} \int_0^{\infty} |F_j(t)|^2 dt.$$

1. Result. For the convolution $F * G$ of $F \in L_p(-\infty, \infty)$ and $G \in L_1(-\infty, \infty)$ ($p \geq 1$), we know the fundamental inequality

$$(1.1) \quad \|F * G\|_p \leq \|F\|_p \|G\|_1.$$  

See, for example, [4, p. 3]. Note that for $F, G \in L_2(-\infty, \infty)$, in general, $F * G \notin L_2(-\infty, \infty)$. In this note, we give

THEOREM 1.1. We take a positive integer $q$ and $F_j \in L_2(0, \infty)$. Then, for the iterated convolution $\prod_{j=1}^{2q} F_j$ of $F_j$, we obtain the inequality

$$(1.2) \quad \int_0^{\infty} \left| \prod_{j=1}^{2q} F_j(t) \right|^2 t^{1-2q} dt \leq \frac{1}{(2q-1)!} \prod_{j=1}^{2q} \int_0^{\infty} |F_j(t)|^2 dt.$$  

Equality holds here if and only if each $F_j$ is expressible in the form $c_je^{-tu}$ for some constant $c_j$ and for some point $u$ such that $Re u > 0$ and $u$ is independent of $j$.

2. Proof of theorem. For $F \in L_2(0, \infty)$ and, in general, $q \geq 2$, we consider the integral transform

$$(2.1) \quad f(z) = \int_0^{\infty} e^{-zt} F(t)t^{q-(1/2)} dt.$$  

Then it is known that $f(z)$ are analytic on Re $z > 0$, the images $f(z)$ form a Hilbert space $H_\nu$ admitting the reproducing kernel

$$(2.2) \quad K_\nu(z, \bar{u}) = \Gamma(2q)/(z + \bar{u})^{2q}$$  

and

$$(2.3) \quad \|f\|^2_{H_\nu} = \int_0^{\infty} |F(t)|^2 dt.$$  

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See, for example, [2, pp. 113–114]. In particular, when \( q = \frac{1}{2} \), we set

\[
(2.4) \quad f_j(z) = \int_0^\infty e^{-zt} F_j(t) \, dt,
\]

and, therefore

\[
(2.5) \quad \|f_j\|^2_{H_{1/2}} = \int_0^\infty |F_j(t)|^2 \, dt.
\]

Note that for an integer \( q \),

\[
(2.6) \quad K_q(z, \bar{u}) = (2q - 1)! K_{1/2}(z, \bar{u})^{2q}.
\]

Hence, from the general theory of Aronszajn [1, pp. 357–362, 350–352], we obtain the inequality

\[
(2.7) \quad \left\| \prod_{j=1}^{2q} f_j \right\|^2_{H_q} \leq \frac{1}{(2q - 1)!} \prod_{j=1}^{2q} (\|f_j\|^2_{H_{1/2}})
\]

as in [3].

On the other hand, we have

\[
(2.8) \quad \prod_{j=1}^{2q} f_j(z) = \int_0^\infty e^{-zt} \left( \prod_{j=1}^{2q} F_j(t) \right) \, dt,
\]

and since \( \prod_{j=1}^{2q} f_j(z) \in H_q \),

\[
(2.9) \quad \prod_{j=1}^{2q} f_j(z) = \int_0^\infty e^{-zt} F^*(t)t^{q-(1/2)} \, dt
\]

for some uniquely determined \( F^* \in L_2(0, \infty) \). From the isometry (2.3) and (2.7), we thus obtain the desired inequality (1.2).

Next, in order to prove the equality statement, we recall that equality holds in (2.7) if and only if each \( f_j \) is expressible in the form \( c_j K_{1/2}(z, \bar{u}) \) for some constant \( c_j \) and for some point \( u \) such that \( \text{Re}\, u > 0 \) and \( u \) is independent of \( j \). See [3] for the case of the unit disc and \( q = 1 \). Note that the result [3] also implies the desired result for, in general, \( q \) for the present situation. See again [1, p. 361, Theorem II].

Hence,

\[
(2.10) \quad f_j(z) = c_j K_{1/2}(z, \bar{u}) = c_j \int_0^\infty e^{-zt} e^{-\bar{u}t} \, dt,
\]

and we thus see that each \( F_j \) is expressible in the desired form \( c_j e^{-t\bar{u}} \).

**References**


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