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A UNIFORMLY, EXTREMELY NONEXTENSIONAL FORMULA OF ARITHMETIC WITH MANY UNDECIDABLE FIXED POINTS IN MANY THEORIES

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ABSTRACT. It is proved that there is a single unary formula F of Peano arithmetic PA and a fixed infinite set $\mathfrak{S}$ of fixed points ϕ of F in PA with the following property. Let T be any recursively enumerable, Σ_1^0-sound extension of PA. Then (i) almost all ϕ in $\mathfrak{S}$ are undecidable in T, and (ii) for all such ϕ and all equivalence relations E satisfying reasonable conditions and refining provable equivalence in T (but not depending on ϕ or T) there is a sentence ψ equivalent to ϕ via E which is not a fixed point of F in T. The theorem furnishes an extreme instance of the difficulties encountered in trying to introduce quantification theory into the diagonalizable algebras of Magari, and yet preserve a central theorem about these structures, the De Jongh-Sambin fixed point theorem. The construction is designed for further applications.

1. Introduction. The diagonalizable algebras were introduced by Magari to study by algebraic methods various self-referential aspects of incomplete theories [**7**, **8**]. (A handy summary is available in Smoryński [**10**].) Let us recall the definition. A diagonalizable algebra (briefly, a da) is a Boolean algebra $\mathfrak{A} = \langle A, +, \cdot, \nu, 0, 1\rangle$ enriched with an additional operator τ such that for all x, y in A, (1) $\tau(1) = 1$, (2) $\tau(x \cdot y) = \tau(x) \cdot \tau(y)$, and (3) $\tau(\tau(x) \to x) \leqslant \tau(x)$, where $x \to y$ is $(\nu x) + y$. These structures constitute a variety $\mathfrak{V}$, and are relevant especially to first order theories T that possess a formula $\dot{T}(x)$ that numerates the theorems of T in Peano arithmetic (PA) (or in R. Robinson's Q) and that satisfies the usual derivability conditions. Thus a typical example of a da consists of the Lindenbaum algebra of the sentences S of such a theory T, with $\tau([S]) = [\dot{T}(\bar{S})]$ (here, we identify a formula A with its Gödel number, and $\bar{A}$ is the numeral of the Gödel number of A; also, $[S]$ is the equivalence class of the sentence S). Of course, essential here is the extensionality of $\dot{T}(x)$; that is, if $T \vdash S_1 \leftrightarrow S_2$, then $T \vdash \dot{T}(\bar{S}_1) \leftrightarrow \dot{T}(\bar{S}_2)$.

The appropriateness of these structures for the stated purposes is confirmed by the De Jongh-Sambin fixed point theorem, which in Sambin's version expresses algebraically a large part at Gödel's diagonalization lemma [**9**, **10**]. (Weaker versions of this theorem were proved independently by C. Bernardi and C. Smoryński [**1**, **2**, **10**].)

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Sambin's form tells us that if p is any polynomial in the operators $+$, $\cdot$, ν, and τ, and x is any variable, every occurrence of which in p falls within the scope of an occurrence of τ, then there effectively can be found a polynomial q that is a fixed point in x of p in every da $\mathfrak{A}$ of $\mathfrak{V}$; that is, $p(q(x_1,\ldots,x_n), x_1,\ldots,x_n) = q(x_1,\ldots,x_n)$ in every da $\mathfrak{A}$ of $\mathfrak{V}$. (A modal logic version of this theorem was independently proved by De Jongh [**10**].) In attempting to generalize these structures by providing an algebraic representation of first order quantification theory and the very source of incompleteness, the nonlogical axioms of a theory, one naturally seeks some counterpart to the characterizing the De Jongh-Sambin theorem. But problems arise as soon as one allows terms in the language that express even simple properties of the proper subject matter of the formal theories (in the case of PA, the natural numbers) that is, even for very simple terms $t(x)$, $\dot{T}(t(x))$ need not be extensional.

Since fixed point theorems such as the De Jongh-Sambin theorem are typically relevant to incompleteness results, in seeking a putative generalization one's interest is naturally focussed on (true) undecidable fixed points of formulas of the form $\neg\dot{T}(t(x))$. Here again it is not difficult to see that for a large number of terms t there are sentences ϕ and ψ, true and undecidable in T, such that $T \vdash \neg\dot{T}(t(\bar{\phi})) \leftrightarrow \phi$, $T \vdash \phi \leftrightarrow \psi$, but $T \nvdash \neg\dot{T}(t(\bar{\psi})) \leftrightarrow \psi$.To remedy this situation, one might try to impose a more stringent condition of extensionality.[2] Thus, one might introduce an equivalence relation E that suitably refines the relation of provable equivalence and work with the refined algebra of sentences consisting of the equivalence classes of the relation E. Alas, for reasonable E this would be to no avail. The purpose of the present note is to demonstrate just how extreme the situation really is. We shall show that there is a single fixed Π_1^0 unary formula F of PA, provably equivalent to $\neg\dot{T}(t(x))$ in a fixed primitive recursive extension PA^+ of PA, t a fixed term of PA^+, which has plenty of arithmetical fixed points ϕ (i.e. such ϕ will be a fixed point of F in PA) with the following property: if T is any one of a large class of recursively enumerable extensions of PA, then almost all true instances of these arithmetical fixed points ϕ are undecidable in T, and not only is $F(x)$ (and hence $\neg\dot{T}(t(x))$) not extensional on ϕ in the sense above, but *on each such* ϕ, $F(x)$ *is weakly extensional* (*in the sense defined below*) *relative to no equivalence relation satisfying certain reasonable conditions*. Thus, $F(x)$ will be a fixed formula of PA that is uniformly nonextensional on many undecidable fixed points of many theories in a very strong sense. Also, our construction is designed so as to lend itself to further applications.

2. Definitions and notation. Our notation is that of [**3** and **4**]. The context in which we work is that of a fixed PR-extension PA^+ of PA, as in [**4**]. If F is a formula of PA^+, F' denotes the formula of PA which is the image of F under the Gödel elimination transformation $'$. (Cf. Feferman [**5**, pp. 52–53] for a brief review of the properties of this transformation.) Borrowing some items from Di Paola [**3**, **4**], we let $\mathfrak{T}(z, x, y)$ be the particular formula there defined that binumerates the Kleene T-predicate in PA; $N_\alpha(z)$ is a recursive function of α and z, and whenever α is an

[2] In this connection, the referee has pointed out the somewhat related considerations of Solovay and Guaspari [**11**] regarding Rosser sentences.

RE-formula numerating the axioms of an re consistent extension T of PA in PA, then for each number z, $W_{N_\alpha(z)} = \{x \mid T \vdash \neg\exists y \mathfrak{T}(\bar{z}, \bar{x}, y)\}$; $\pi(\alpha, m, n)$ is a recursive function such that when α is an RE-formula as above, then for all m, n such that $W_m \cap W_n = \varnothing$, $W_m = \{x \mid T \vdash \exists y \mathfrak{T}(\bar{z}_0, \bar{x}, y)\}$ and $W_n = W_{N_\alpha(z_0)}$, where $z_0 = \pi(\alpha, m, n)$; and $W_m = W_{z_0}$ if T is weakly ω-consistent (i.e., Σ_1^0-sound) [**3**, Theorem 6].

We have need of a few more definitions. Let T be a theory and E an equivalence relation on the sentences of T which is a refinement of provable equivalence in T, i.e., $\phi E \psi \Rightarrow T \vdash \phi \leftrightarrow \psi$. Let F be a unary formula of T. F is said to be *extensional on* ϕ *relative to* E if for all sentences ψ of T, $\phi E \psi \rightarrow F(\bar{\phi}) E F(\bar{\psi})$; F is *extensional relative to* E if E is extensional on every sentence ϕ relative to E; F is *weakly extensional on* ϕ *relative to* E if for all ψ, $\phi E \psi \Rightarrow T \vdash F(\bar{\phi}) \leftrightarrow F(\bar{\phi})$; F is *weakly extensional relative to* E if for all ϕ, F is weakly extensional on ϕ relative to E.

(*) For each theory T, let E_T be an equivalence relation on the sentences of T having the following properties:

(i) E_T is re (recursively enumerable) as a binary relation;

(ii) for each ϕ, $[\phi]_{E_T}$, the equivalence class of ϕ relative to E_T, is infinite if ϕ contains quantifiers;

(iii) E_T is a refinement of provable equivalence in T.

Note 1. E_T does not depend on T. Property (iii) relates E_T to T, but this is not a relation of dependence. For example, we may take T to be ZFC and E_T to be provable equivalence in Robinson's Q [**11**]. Also, in (ii) it is allowed, but not required, that $[\phi]_{E_T}$ be finite if all members of $[\phi]_{E_T}$ are quantifier-free; it is required that $[\phi]_{E_T}$ be infinite, if some member of $[\phi]_{E_T}$ contains quantifiers.

3. The Theorem.

THEOREM. *Let α be an RE-formula numerating the axioms of PA in PA. Then there is a fixed Π_1^0 formula $F_\alpha(x)$ of PA with the following properties*:

(i) *There is an infinite recursive set $\mathfrak{F}$ of fixed points of $F_\alpha(x)$ in PA and the set $\mathfrak{E} = \{\phi \mid \phi \in \mathfrak{F}$ and $\omega \models F_\alpha(\bar{\phi})$, ω the standard model of PA$\}$ is not recursive*;

(ii) *For each recursively enumerable, Σ_1^0-sound extension T of PA and almost all ϕ in $\mathfrak{E}$ (i.e. all but a finite number), ϕ is undecidable in T*;

(iii) *For all E_T as in* (*) *above with T as in* (ii) *and all $\phi \in \mathfrak{E}$, there is a sentence ψ of T such that $\phi E_T \psi$, but ψ is not a fixed point of F_α in T. (Thus, for all such E_T, F_α is not even weakly extensional on any ϕ in $\mathfrak{E}$, and hence not weakly extensional on many arithmetical fixed points ϕ undecidable in T.)*

Moreover, there is a fixed term $t(x)$ of PA^+ such that

$$\mathrm{PA}^+ \vdash \neg\dot{\mathrm{T}}\mathrm{hm}_\alpha(t(x)) \leftrightarrow F_\alpha(x)$$

where $\dot{\mathrm{T}}\mathrm{hm}_\alpha$ is an RE-formula obtained from α as in Feferman [**5**] *that numerates the theorems of PA in PA. Here, and in the sequel, $\dot{\mathrm{T}}\mathrm{hm}_\alpha(x)$ corresponds to the $\dot{T}(x)$ of the Introduction, specifically to the $\dot{T}(x)$ of PA.*

Note 2. The ϕ of (ii) and the E_T of (iii) may be utterly independent of one another. In general, nothing depends on E_T except the ψ of (iii).

PROOF. Let us put

$$H(u, z, x) = \neg\dot{\mathrm{T}}\mathrm{hm}_\alpha\left(\dot{S}b\left(\overline{\neg\exists y\mathfrak{T}(z, x, y)}\,\middle|\,\begin{matrix}\dot{z} & \dot{x} & & \dot{z} & \dot{x}\\ n\dot{m}_z & n\dot{m} & \dot{S}b(u\,| & & \\ & & & n\dot{m}_z & n\dot{m}_x)\end{matrix}\right)\right)'.$$

By the fixed point theorem for formulas, there is a formula $G(z, x)$ such that

$$\mathrm{PA} \vdash G(z, x) \leftrightarrow H(\overline{G}, z, x).$$

Now, for all numbers z and x

$$\mathrm{PA}^+ \vdash \dot{S}b\left(\overline{\neg\exists y\mathfrak{T}(z, x, y)}\,\middle|\,\begin{matrix}\dot{z} & \dot{x} & & \dot{z} & \dot{x}\\ \dot{n}m_{\bar{z}} & \dot{n}m & Sb(\overline{G}\,| & & \\ & & & n\dot{m}_{\bar{z}} & n\dot{m}_{\bar{x}})\end{matrix}\right)$$

$$\approx \overline{\neg\exists y\mathfrak{T}\left(\bar{z}, \overline{G(\bar{z}, \bar{x})}, y\right)}.$$

Thus, for all numbers z and x

$$\mathrm{PA} \vdash G(\bar{z}, \bar{x}) \leftrightarrow \neg\dot{\mathrm{T}}\mathrm{hm}_\alpha\left(\overline{\neg\exists y\mathfrak{T}\left(\bar{z}, \overline{G(\bar{z}, \bar{x})}, y\right)}\right).$$

So, by Di Paola [**3**, **4**],

$$PA \vdash G(\bar{z}, \bar{x}) \leftrightarrow \neg\exists y\mathfrak{T}\left(\overline{N_\alpha(z)}, \overline{G(\bar{z}, \bar{x})}, y\right).$$

For each number z, we put $\mathcal{G}_z = \{G(\overline{\pi(\alpha, r, z)}, \bar{x})\}$ where x varies, and r is an index of the set $\mathfrak{R}$ of numbers that are not sentences of PA (with a typical Gödel numbering, $\mathfrak{R}$ is an infinite recursive set): $W_R = \mathfrak{R}$. For each z, $\mathcal{G}_z$ is an infinite recursive set, as is the set $\omega - (\mathcal{G}_z \cup \mathfrak{R})$.

Let us observe that, by the proof of the fixed point theorem, for every z, every member of $\mathcal{G}_z$ contains quantifiers. By, for example, Post's construction of a simple set, we see that there are recursive functions k and h such that for each z, $W_{k(z)}$ is simple in $\omega - (\mathcal{G}_z \cup \mathfrak{R})$ and $W_{h(z)}$ is simple in $\mathcal{G}_z$. Let g be a recursive function such that for each z, $W_{g(z)} = W_{k(z)} \cup W_{h(z)}$. Thus, for each z, $W_{g(z)}$ is simple in each of $\mathcal{G}_z$ and $\omega - (\mathcal{G}_z \cup \mathfrak{R})$, and also simple in $\mathcal{G}_z \cup (\omega - (\mathcal{G}_z \cup \mathfrak{R}))$, the set of sentences of PA. By the recursion theorem, there can be found a z_0 such that $W_{z_0} = W_{g(z_0)}$. But $W_{g(z_0)} \cap \mathfrak{R} = \varnothing$, so by Theorem 6 of [**3**] stated above, $W_{z_0} = W_{N_\alpha(\pi(\alpha,r,z_0))}$; that is $W_{N_\alpha(\pi(\alpha,r,z_0))} = W_{g(z_0)}$ and $W_{\pi(\alpha,r,z_0)} = W_r$.

Now, let T be any re, Σ_1^0-sound extension of PA. Since $W_{N_\alpha(\pi(\alpha,r,z_0))} = W_{z_0} = W_{g(z_0)}$ and $W_{g(z_0)}$ is not recursive, $\mathcal{G}_{z_0} - W_{g(z_0)}$ is infinite, so there is a number x_0 such that

$$\neg\exists y\mathfrak{T}\left(\overline{N_\alpha(\pi(\alpha, r, z_0))}, \overline{G\left(\overline{\pi(\alpha, r, z_0)}, \bar{x}_0\right)}, y\right)$$

is undecidable in T, and true in the standard model of T. (Note the use here of the Σ_1^0-soundness of T.) This follows without use of the fact that $W_{g(z_0)}$ is simple in $\mathcal{G}_{z_0}$.

Using this latter fact and that

$$\left\{G\left(\overline{\pi(\alpha, r, z_0)}, \bar{x}\right) \mid T \vdash \neg\exists y \mathfrak{T}\left(N_\alpha\left(\overline{\pi(\alpha, r, z_0)}\right), \overline{G\left(\overline{\pi(\alpha, r, z_0)}, \bar{x}\right)}, y\right)\right\} \subset \mathfrak{G}_{z_0}$$

and that the function $G(\overline{\pi(\alpha, r, z_0)}, \bar{x})$ is 1-1 in x, we see that for all but a finite number of x such that $G(\overline{\pi(\alpha, r, z_0)}, \bar{x}) \in \omega - W_{z_0}$, $G(\overline{\pi(\alpha, r, z_0)}, \bar{x})$ is true and undecidable in T. Thus, putting $F_\alpha(x) = \neg\exists y \mathfrak{T}(\overline{N_\alpha(\pi(\alpha, r, z_0))}, x, y)$, x now a free variable, and $\mathfrak{F} = \mathfrak{G}_{z_0}$, we see that (i) and (ii) of the theorem are proven.

Let E_T be any equivalence relation satisfying the conditions stated just prior to the statement of the theorem. We put $S_\phi = \{\psi \mid \phi E_T \psi, \phi = G(\overline{\pi(\alpha, r, z_0)}, \bar{x})$ for some x, and ϕ undecidable in $T\}$. Since by hypothesis E_T is defined on sentences of T and $[\phi]_{E_T}$ is infinite and re, and W_{z_0} is simple in the set of sentences of T, there is a $\psi \in S_\phi \cap W_{z_0}$, so that $\mathrm{PA} \vdash \neg F_\alpha(\bar{\psi})$. Since ϕ is undecidable in T, we have that $T \nvdash F_\alpha(\bar{\psi}) \leftrightarrow \psi$. Thus, (iii) is proved.

If we take the term $t(x)$ of PA^+ as follows,

$$t(x) = \dot{S}b\left(\overline{\neg\exists y \mathfrak{T}\left(\overline{\pi(\alpha, r, z_0)}, x, y\right)}\Big|^{\dot{x}}_{\dot{n}m_x}\right),$$

we see that $\mathrm{PA}^+ \vdash \neg\dot{\mathrm{T}}\mathrm{hm}_\alpha(t(x)) \leftrightarrow F_\alpha(x)$. $\square$

Observation. To illustrate the theorem, let us consider the logic of PA, that is, the predicate calculus with equality on the first order language $\mathfrak{L}(\sigma)$ having signature $\sigma = \langle 0, +, \cdot, S\rangle$. As in Kleene [**6**], let us say that a formula A is congruent to a formula B if, to put it briefly, A and B are symbol by symbol the same formal expression except that they may differ in their bound variables. Congruent formulas are provably equivalent by means of logical provability alone. We define the following equivalence relation E on the sentences of $\mathfrak{L}(\sigma)$: $\phi E \psi \Leftrightarrow \phi$ and ψ are congruent sentences of $\mathfrak{L}(\sigma)$. Thus, for all re, Σ_1^0-sound extensions T of PA. E is an E_T as defined above. Now, let us take T to be ZFC. (As usual, we ignore the linguistic distinctions between PA and the subtheory of ZFC that is equivalent to PA.) Then by the theorem almost all members of $\mathfrak{S}$ are fixed points ϕ of F_α in PA that are true and undecidable in ZFC; and for any such ϕ there is a ψ congruent to ϕ such that ψ is not a fixed point of F_α in ZFC.

We may make the equivalence relation E so that ϕ and ψ bear a still stronger resemblance to one another. For example, we may define E so that $\phi\, E \psi$ if and only if ϕ and ψ are congruent sentences of $\mathfrak{L}(\sigma)$, and if ϕ contains quantifiers, then ϕ and ψ differ at most in the variable bound by the left-most quantifier of ϕ and ψ. Thus, by means of a revision, in the light of the theorem of Matiyasevič, Davis, Putnam and Robinson that all re sets are Diophantine and the results in [**3**, **4**] cited in our proof, a careful inspection of the proof reveals that the undecidable fixed points ϕ of F_α may be taken to be of the form

$$\forall x_1 \forall x_2 \cdots \forall x_n\left[P\left(\bar{k}, \bar{m}, \bar{x}, x_1, x_2, \ldots, x_n\right) \not\approx Q\left(\bar{k}, \bar{m}, \bar{x}, x_1, x_2, \ldots, x_n\right)\right],$$

where k, m, and x are numbers, and P and Q are fixed polynomials of PA; that is, every ϕ in $\mathfrak{S}$ is obtainable by substituting a number x for the variable v in the polynomial inequality $P(\bar{k}, \bar{m}, v, x_1, x_2, \ldots, x_n) \not\approx Q(\bar{k}, \bar{m}, v, x_1, x_2, \ldots, x_n)$. In the

example of the theorem under discussion, the associated ψ that is not a fixed point of F_α is of the form

$$\forall u \forall x_2, \ldots, \forall x_n \left[P(\bar{k}, \bar{m}, \bar{x}, u, x_2, \ldots, x_n) \neq Q(\bar{k}, \bar{m}, \bar{x}, u, x_2, \ldots, x_n) \right],$$

where u is some variable distinct from x_1.

Of course, the strength of the theorem lies in the fact that the formula F_α and the set $\mathcal{F}$ of fixed points of F_α remain invariant, and yet do the job for *all* re, Σ_1^0-sound extensions T of PA and *all* equivalence relations E_T of the aforesaid kind.

Note 3. Why do we take $W_{g(z_0)}$ to be simple not only in the set of sentences of T, but also separately in $\mathcal{G}_{z_0}$ and $\omega - (\mathcal{G}_{z_0} \cup \mathcal{R})$? If $W_{g(z_0)}$ were simple only in the set of sentences of T, then $W_{g(z_0)} \cap \mathcal{G}_{z_0}$ could be recursive, and we could not guarantee that for all re, Σ_1^0-sound extensions T of PA, $F_\alpha(x)$ have arithmetical fixed points ϕ that are undecidable in T. If $W_{g(z_0)}$ were simple only in $\mathcal{G}_{z_0}$, then possibly $S_\phi - \{\phi\} \subset \omega - (\mathcal{G}_{z_0} \cup \mathcal{R})$ and thus $S_\phi \cap W_{z_0} = \varnothing$, rendering impossible the conclusion that $T \vdash \neg F_\alpha(\bar{\psi})$ for some $\psi \in S_\phi$.

Note 4. It is in part our desire to achieve uniformity which has required us to include the condition of Σ_1^0-soundness among the hypotheses of the theorem, i.e. our objective of finding a single Π_1^0 formula $F_\alpha(x)$ with the stated properties. For our $F_\alpha(x)$ the necessity that the re extension T of PA be Σ_1^0-sound is clear. Let T be the re, consistent, not Σ_0^1-sound extension of PA obtained by adding to PA as new axioms the sentences $\neg F_\alpha(\bar{x})$ for all numbers x. Obviously, our theorem does not apply to T. The existence of a formula having the properties of our $F_\alpha(x)$ with α a natural binumeration of PA (i.e. α an RE-formula), but uniform for all consistent re extensions of PA remains an open question.

On another point, since for all ϕ undecidable in an re, Σ_1^0-sound extension T of PA (as in (ii) of the theorem), the number of sentences ψ provably equivalent to ϕ in T that are fixed points of our F_α is finite. The equivalence relations E_T need not be re, so long as they satisfy the remaining conditions stated in §2. This was noticed by C. Bernardi after a reading of our proof. Of course, if E_T is not re, one forgoes the possibility of effectively obtaining a ψ that is not a fixed point of F_α in T from a ϕ that is equivalent to ψ via E_T. In our proof, such ψ are obtained from the pertinent ϕ by a uniform procedure, that is, as a recursive function of α, T, E_T and ϕ.

Finally, we observe that our construction lends itself to other applications with but minor changes. In particular, the use of a simple set is governed only by the particular question under investigation in this note. If one confines oneself to simple sets from the beginning, the argument may be somewhat simplified. But by the construction there is a fixed formula $A(z, y)$ of PA with two free variables, and for each number z the formula $G(\bar{z}, \bar{x})$ is a fixed point of $A(\bar{z}, y)$ in PA for every x, and $\mathrm{PA}^+ \vdash A(\bar{z}, y) \leftrightarrow \neg \mathrm{Thm}_\alpha(t_z(y))$ for a suitable term t_z. If one wishes to introduce a creative set, or a pair of re effectively inseparable sets, or some combination of diverse re sets, one has only to use the separating function $\pi(\alpha, m, n)$ and the recursion theorem as in the proof to obtain a w such that $\neg A(\bar{w}, y)$ numerates in PA a creative set, or the pair of re effectively inseparable sets etc., and $A(\bar{w}, y)$ has the set $\mathcal{K}_W = \{G(\bar{w}, \bar{x}) \mid \text{all } x\}$ of sentences as fixed points in PA. Undecidability

properties of the fixed points depend on which type of re set one has specified. The formula $A(z, y)$ and the set $\mathcal{H} = \{G(\bar{z}, \bar{x})\}$ of fixed points are defined *a priori once and for all*. One may say that one has a surface $\mathcal{H} = \{G(\bar{z}, \bar{x})\}$ in the (z, x)-plane, and for suitable w selects a curve $\mathcal{H}_W = \{G(\bar{w}, \bar{x})\}$ of fixed points of $A(\bar{w}, y)$ on $\mathcal{H}$, the choice depending upon which application one has in mind. We hope to explore this situation further.

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