3-MANIFOLDS WHICH CONTAIN NONPARALLEL PROJECTIVE PLANES

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ABSTRACT. We show that if a closed, connected 3-manifold with a Heegaard splitting of genus three contains mutually disjoint and nonparallel 2-sided projective planes, then the manifold is homeomorphic to the connected sum of $P^2 \times S^1$ and the twisted 2-sphere bundle over the circle.

1. Introduction. In [7] Ochiai has shown that if a 3-manifold with a Heegaard splitting of genus two contains a 2-sided projective plane then it is homeomorphic to $P^2 \times S^1$. Negami [5] has shown that there exist infinitely many 3-manifolds, each of which has a Heegaard splitting of genus three and contains a 2-sided projective plane. In this paper we show that a 3-manifold with a Heegaard splitting of genus three and two nonparallel 2-sided projective planes is unique.

THEOREM 2. Let $M$ be a closed, connected 3-manifold with a Heegaard splitting of genus three. Assume $M$ contains two mutually disjoint and nonparallel 2-sided projective planes. Then $M$ is homeomorphic to the connected sum $P^2 \times S^1 \# K$, where $K$ denotes the twisted 2-sphere bundle over the circle.

We note that there are infinitely many irreducible, closed 3-manifolds, each of which contains mutually disjoint and nonparallel 2-sided projective planes (Negami [6], Row [9]).

Let $M$ be a closed, connected, prime 3-manifold. Then as consequences of Theorem 2 we have

COROLLARY 1. If $M$ contains two mutually disjoint and nonparallel 2-sided projective planes, then the Heegaard genus of $M$ is greater than three.

COROLLARY 2. If the minimal number of generators of $\pi_2(M)$ is greater than one, then the Heegaard genus of $M$ is greater than three.

We work throughout in the piecewise linear category. For definitions of Heegaard splitting and other standard terms in three-dimensional topology, we refer to [2].

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2. Theorem 1. In this section, for the proof of Theorem 2, we will slightly modify Ochiai's result [8] as Theorem 1.

THEOREM 1. Let $M$ be a closed, connected, prime 3-manifold with a Heegaard splitting $(V_1, V_2)$. Assume $M$ contains 2-sided projective planes $P_1, \ldots, P_n$ ($n \geq 2$)
which are mutually disjoint and nonparallel. Then there is an ambient isotopy $h_t \ (0 \leq t \leq 1)$ of $M$ such that $h_1(P_i) \cap V_1, \ldots, h_1(P_n) \cap V_1$ are mutually nonparallel meridian disks of $V_1$.

Lemma 2.1. Let $M, (V_1, V_2), P_1, \ldots, P_n$ be as in Theorem 1. Then there is an ambient isotopy $g_t \ (0 \leq t \leq 1)$ of $M$ such that $g_t(P_i) \cap V_1 \ (i = 1, \ldots, n)$ is a disk.

Proof. This can be proved by the argument of inverse operations of an isotopy of type A [3] defined in [8]. In [8] Ochiai considered one projective plane but this argument applies to finitely many mutually disjoint 2-sided projective planes.

Lemma 2.2. Let $H$ be a solid Klein bottle and $Q$ a 2-sided Möbius band properly embedded in $H$. If we attach a 2-handle to $H$ along $\partial Q$ then we get $P^2 \times I$.

Proof. Let $D$ be a meridian disk of $H$, i.e. $D$ cuts $H$ into a 3-cell. By [4] we may suppose $\partial D$ intersects $\partial Q$ transversely in two points. So we may suppose $Q \cap D$ consists of an arc and some simple loops. Since $H$ is irreducible we can move $D$ by an isotopy so that $Q \cap D$ consists of an arc. Then $H$ is homeomorphic to $Q \times I$, where $Q$ corresponds to $Q \times \{1/2\}$, and we have the conclusion of Lemma 2.2.

Proof of Theorem 1. By Lemma 2.1 there is an ambient isotopy $g_t$ of $M$ such that $D_i := g_t(P_i) \cap V_1 \ (i = 1, \ldots, n)$ is a disk. If $D_i$ separates $V_1$ then $P_i$ separates $M$ into $M_1$ and $M_2$. Let $D(M_1)$ be a double of $M_1$. By Poincaré duality we see that $\chi(D(M_1)) = 0$. On the other hand, we easily see that $\chi(D(M_1)) = 2\chi(M_1) - \chi(M_1) = 2\chi(M_1) - 1$. Hence $\chi(M_1) = 1/2$, a contradiction.

Assume that some $D_i$ and $D_j$, say $D_1$ and $D_2$, are parallel in $V_1$. There is an annulus $A$ in $F = \partial V_1$ such that $A \cap (D_1 \cup D_2) = \partial A = \partial D_1 \cup \partial D_2$. Let $E = A \cup (V_2 \cap P_1) \cup (V_2 \cap P_2)$. $E$ is a 2-sided Klein bottle in $V_2$. By the loop theorem [2] and irreducibility of $V_2$, we see that $E$ bounds a solid Klein bottle $H$ in $V_2$. On the other hand, $A \cup D_1 \cup D_2$ bounds a 3-cell $C$ in $V_1$. By Lemma 2.2 we get $P^2 \times I$ by attaching $C$ to $H$ along $A$, which contradicts the fact that $P_1$ and $P_2$ are not parallel.

This completes the proof of Theorem 1.

3. Theorem 2. Let $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ and $D^2 = \{z \in \mathbb{C}: |z| \leq 1\}$.

We get $P^2 \times S^1$ from $D^2 \times S^1$ by identifying its boundary points in the following manner:

$$(z_1, z_2) \sim (-z_1, z_2) \quad (z_1 \in \partial D^2).$$

Let $l_1 = \{0\} \times S^1$ be a simple loop in $P^2 \times S^1$, and $l_2$ an essential, simple loop in $P^2 = D^2 \times \{1\}/\sim$ which intersects $l_1$ in a single point. We note that $l_2$ is unique up to an isotopy of $P^2$. Let $K_1 = \text{cl}(P^2 \times S^1 - N(l_2))$, $V_1 = N(l_1 \cup l_2)$, $V_2 = \text{cl}(P^2 \times S^1 - V_1)$, where $N(X)$ denotes a regular neighborhood of a polyhedron $X$. Then we easily see that $(V_1, V_2)$ is a genus two Heegaard splitting of $P^2 \times S^1$.

Lemma 3.1. Let $P_1, P_2$ be the components of $\partial(P^2 \times I)$ and $D_i \ (i = 1, 2)$ a disk in $P_i$. If $M$ is obtained from $P^2 \times I$ by identifying $D_1$ and $D_2$, then $M$ is homeomorphic to $K_1$.

Proof. We give $D_1$ and $D_2$ fixed orientations. Then we have two possibilities when we attach $D_1$ to $D_2$ depending on whether the two orientations coincide or
differ. But if we slide $D_1$ along an orientation reversing loop in $P_1$, we get the same manifolds by each of the above attachings.

Let $D = P^2_1 - N(l_2)$. Then $D$ is a disk properly embedded in $K_1$ and $D$ cuts $K_1$ into $P^2 \times I$. Hence $M$ is homeomorphic to $K_1$.

**Lemma 3.2.** Let $V$ be a genus two handlebody and $Q$ a nonseparating 2-sided Möbius band properly embedded in $V$. If $M$ is obtained from $V$ by attaching a 2-handle along $\partial Q$, then $M$ is homeomorphic to $K_1$.

**Proof.** We easily find a disk $D$ in $V$ such that $D \cap \partial V = \partial D \cap \partial V = \alpha$ is an arc, $D \cap Q = \partial D \cap Q = \beta$ is an essential arc of $Q$ and $\alpha \cup \beta = \partial D$, $\alpha \cap \beta = \partial \alpha = \partial \beta$. By performing surgery on $Q$ along $D$ we get a disk $D'$ properly embedded in $V$. Since $Q$ is nonseparating in $V$, $D'$ is a meridian disk. Since $Q$ is 2-sided, we can move $D'$ by a small isotopy so that $D' \cap Q = \emptyset$. Let $V'$ be $V$ cut along $D'$. $V'$ is a solid Klein bottle, for it contains a 2-sided Möbius band $Q$. By Lemma 2.2 we get $P^2 \times I$ by attaching a 2-handle to $V'$ along $\partial Q$. Let $D_1$, $D_2$ be the copies of $D'$ on $\partial V'$. Since $Q$ is nonseparating in $V$, $D_1$ and $D_2$ are contained in mutually distinct components of $\partial (P^2 \times I)$. Hence by Lemma 3.1, $M$ is homeomorphic to $K_1$.

**Proposition.** Let $M$ be a closed, connected 3-manifold which has a Heegaard splitting $(V_1, V_2)$ of genus three. Assume there are mutually disjoint and nonparallel 2-sided projective planes $P_1$, $P_2$ in $M$ such that $D_1 = P_1 \cap V_1$ (resp. $D_2 = P_2 \cap V_2$) is a disk and $Q_1 = P_1 \cap V_2$ (resp. $Q_2 = P_2 \cap V_1$) is a Möbius band. Then $M$ is homeomorphic to $P^2 \times S^1 \# K$, where $K$ is the twisted 2-sphere bundle over the circle.

**Proof.** Since $P_i$ ($i = 1, 2$) is nonseparating in $M$, $D_1$ (resp. $D_2$) is a meridian disk of $V_1$ (resp. $V_2$).

Then we claim that $D_1 \cup Q_2$ (resp. $D_2 \cup Q_1$) does not separate $V_1$ (resp. $V_2$). Assume $D_1 \cup Q_2$ separates $V_1$. If $D_2 \cup Q_1$ does not separate $V_2$, we can find a loop $l$ in $\partial V_2$ such that $l$ intersects $\partial D_2 \cup \partial Q_1$ transversely in a single point, which contradicts the fact that $D_1 \cup Q_2$ separates $V_1$. So $D_2 \cup Q_1$ also separates $V_2$. Let $V'$ be $V_1$ cut along $D_1$, and $D'_1$, $D''_1$ the copies of $D_1$ in $\partial V'$. Then $V'$ is a genus two handlebody which contains a 2-sided Möbius band $Q_2$. Then we have two cases.

**Case 1.** $Q_2$ is parallel to a Möbius band $Q'$ in $\partial V'$. In this case $D'_1$ or $D''_1$ is contained in $Q'$, for if not then $P_2$ is isotopic into $V_2$, a contradiction. Moreover, since $Q_2$ is nonseparating in $V_1$, $Q'$ contains only one of $D'_1$ and $D''_1$. Hence $P_2 \cup D_1$ cuts $V_1$ into a solid Klein bottle and the other component.

**Case 2.** $Q_2$ is not boundary parallel.

In this case by the argument of the proof of Lemma 1 of [7], we see that there is a complete system of meridian disks $\{D', D''\}$ of $V'$ such that $D' \cap Q_2 = \emptyset$ and $D'' \cap Q_2$ is an essential arc of $Q_2$. We may suppose

\[(D' \cup D'') \cap (D'_1 \cup D''_1) = \emptyset.\]

$D'$ cuts $V'$ into a solid Klein bottle $V''$ for it contains a 2-sided Möbius band $Q_2$. Then, by the proof of Lemma 2.2, $Q_2$ cuts $V''$ into two solid Klein bottles. Since $D_1 \cup Q_2$ separates $V_1$, the two copies of $D_1$ are on mutually distinct components of
V" cut along Q2, and the two copies of D' are on the same component of V" cut along Q2. Hence D1 U Q2 cuts V1 into a solid Klein bottle and the other component.

So in either case D1 U Q2 cuts V1 into a solid Klein bottle R1 and the other component. By the same argument D2 U Q1 cuts V2 into a solid Klein bottle R2 and the other component. Let D'2, Q'1 (resp. D', Q2) be the copies of D2, Q1 (resp. D1, Q2) on ∂R1 (resp. ∂R2). By considering the Euler characteristic we see that cl(∂R1 - (D'2 U Q'1)) and cl(∂R2 - (D'1 U Q2)) are identified in M. By Lemma 2.2 we get P2 x I by attaching N(D'2) (resp. N(D'1)) to R2 (resp. R1) and so we get S2 xI by attaching R1 to R2 along cl(∂R1 - (D'2 U Q'1)) and cl(∂R2 - (D'1 U Q2)). But this contradicts the fact that P1 and P2 are not parallel.

Hence D1 U Q2 (resp. D2 U Q1) does not separate V1 (resp. V2) and the claim is established.

Let V1' (resp. V2') be V1 cut along D1 (resp. V2 cut along D2) and K1' (resp. K2') be the manifold obtained by attaching N(D1) (resp. N(D2)) to V1' (resp. V1). By Lemma 3.2 and the above claim we see that each of K1' and K2' is homeomorphic to K1. Let D be a disk in K1, defined in the proof of Lemma 3.2, and D' (resp. D'') the corresponding disk in K1' (resp. K2'). Then by [4], ∂D' and ∂D'' are isotopic in K1' ∩ K2'' = ∂K1' = ∂K2''. So we may suppose ∂D' and ∂D'' are identified in M and then D' U D'' is a nonseparating 2-sphere in M. So by Lemmas 3.8 and 3.17 of [2], M = M1 #K. By Corollary II.10 of [3], M1 has a Heegaard splitting of genus two, and by using a cut and paste method on P1 we see that M1 contains a 2-sided projective plane. Hence by [7], M1 is P2 x S1 and this completes the proof of the Proposition.

PROOF OF THEOREM 2. Let (V1, V2) be a Heegaard splitting of genus three of M. If M is prime then by Theorem 1 we may suppose P1 ∩ V1 (i = 1, 2) is a disk. Then by performing an isotopy of type A [3] on P1, i = 1 or 2, say 2, we can move P1, P2 to the position as in the Proposition. If M is not prime, then by Corollary II.10 of [3] and [7], M is either P2 x S1 #K or P2 x S1 #L_n, where L_n denotes a three-dimensional lens space. We note that the orientable three-manifold with a Heegaard splitting of genus one is either a lens space or S^2 x S^1, and the nonorientable 3-manifold with a Heegaard splitting of genus one is K [2, 4]. Assume M is P2 x S1 #L_n. Then there is a 2-sphere S in M such that S cuts M into P2 x S1 - Int B_1 and L_n - Int B_2, where B_1 (resp. B_2) is a 3-cell in P2 x S1 (resp. L_n). Since P2 x S1 and L_n are irreducible, there is an ambient isotopy g_t (0 ≤ t ≤ 1) of M such that

\[ g_1(P_1) \subset P2 x S1 - Int B_1^3 \quad (i = 1, 2). \]

Hence g1(P1) and g2(P2) are parallel in M, a contradiction. So M is either P2 x S1 #K or P2 x S1 #S2 x S1. But by Lemma 3.17 of [2] these are pairwise homeomorphic. This completes the proof of Theorem 2.

PROOF OF COROLLARIES. Corollary 1 is an immediate consequence of Theorem 2. So we will prove Corollary 2. Suppose the minimal number of generators of π2(M) is n (≥ 2). By the projective plane theorem [1] there is a system of mutually disjoint 2-spheres and 2-sided projective planes \{Q_1, \ldots, Q_n\} which represents a generator of π2(M). Assume that some Q_i is a 2-sphere. Since M is prime, by Lemma 3.13 of [2] we see that M is a 2-sphere bundle over a circle. But this contradicts that n ≥ 2. Hence each Q_i is a projective plane. Clearly \{Q_1, \ldots, Q_n\}
are mutually nonparallel. So by Corollary 1 the Heegaard genus of $M$ is greater than three.

REFERENCES


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