NILPOTENCY OF DERIVATIONS. II
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ABSTRACT. The authors recently proved that for a semiprime ring without 2-torsion, a nilpotent derivation must have odd nilpotency. In this paper, we show the intriguing phenomenon that for a semiprime ring with characteristic 2, the nilpotency of a nilpotent derivation must be of the form $2^n$. Combining these two results, we show that for a general semiprime ring with no torsion condition, the nilpotency of a nilpotent derivation is either odd or a power of 2.

Let $R$ be a semiprime ring and $\partial$ a derivation of $R$. $\partial$ is said to be nilpotent if $\partial^m R = (0)$ for some positive integer $m$. The smallest such $m$ is called the nilpotency of $\partial$. It was proved in [1] that the nilpotency of $\partial$ is always odd if $R$ is 2-torsion free. The purpose of this note is to settle the general case without any torsion condition on $R$. It is known that the nilpotency of $\partial$ is either a power of 2 or an odd number. This result will be used in [2] to show the invariance of the nilpotency of a derivation on any nonzero ideals.

Let us first consider the complementary case that $R$ is of characteristic 2.

**Theorem 1.** Let $R$ be a semiprime ring of characteristic 2. Suppose $\partial$ is a derivation of $R$ and $\partial^m = 0$ where $2^k < m < 2^{k+1} - 1$ for some positive integer $k$. Then $\partial^{2^k} = 0$ and hence the nilpotency of $\partial$ is a power of 2.

**Proof.** We proceed by induction on $k$. If $k = 1$, then $m = 3$ and $\partial^3 = 0$. For any $x, y \in R$, $0 = \partial^3(\partial xy) = \partial^2 x \partial^2 y$. By replacing $y$ by $yx$ and noting that $\partial^2$ is also a derivation of $R$, we obtain $0 = \partial^2 x \partial^2 (yx) = \partial^2 x (\partial^2 y x + y \partial^2 x) = \partial^2 xy \partial^2 x$ and hence by the semiprimeness of $R$, $\partial^2 x = 0$ for all $x \in R$.

We now assume $k > 1$.

**Case 1.** Suppose $m < 2^{k+1} - 1$. Then $m = \sum_{j=0}^{k} \alpha_j 2^j$ where each $\alpha_j$ is either 0 or 1 and at least one of the $\alpha_j$’s is zero. Let $i$ be the smallest one with $\alpha_i = 0$. If $i = 0$, then $m = 2m_0$ where $m_0 = \sum_{j=1}^{k} \alpha_j 2^{j-1}$ and $2^{k-1} < m_0 < 2^{k} - 1$. Set $\delta = \partial^2$. Then $\delta$ is a derivation with $\delta^{m_0} = 0$. By the induction hypothesis, $\delta^{2^{k-1}} = 0$ or $\partial^{2^k} = 0$. Now assume $i > 0$. Then $\alpha_0 = 1$ and $m + 1 = 2^i + \sum_{j=i+1}^{k} \alpha_j 2^j = 2^n n$ where $n = 1 + \sum_{j=i+1}^{k} \alpha_j 2^{j-i}$. Let $\delta = \partial^{2^i}$. Then $\delta$ is a derivation of $R$ and $2^{k-1} < n \leq 2^{k+1} - 1$. Again by induction hypothesis, $\delta^{2^{k-1}} = 0$ or $\partial^{2^k} = 0$.

**Case 2.** Suppose $m = 2^{k+1} - 1$. In view of Case 1, we need only show that $\partial^{m-1} = 0$. Suppose to the contrary that $\partial^{m-1} \neq 0$. Then since, for any $x, y \in R$, $0 = \partial^m(\partial^{m-2} xy) = \partial^{m-1} x \partial^{m-1} y$, there exists $a \in \partial^{m-1} R$, $a \neq 0$, such that $\partial a = 0$ and $\partial^{m-1} xa = a \partial^{m-1} y = 0$ for all $x, y \in R$. Let $I$ be the ideal of $R$ generated by $\partial^{m-1} R$ and let $S = \{(s, t)\mid s$ and $t$ are positive integers such that .
Let $b \in I$ with $b \neq 0$, $\partial b = 0$, $\partial^g Rb = b \partial^g R = (0)$}. Clearly $(m - 1, m - 1) \in S$.

Partially order $S$ the following way: $(s_1, t_1) < (s_2, t_2)$ iff $s_1 \leq s_2$ and $t_1 \leq t_2$. Let $(p, q)$ be a minimal element in $S$ and $c$ be a nonzero element in $I$ such that $\partial c = 0$ and $\partial^p Rc = c \partial^q R = (0)$.

If one of $p$ and $q$ is $\leq 2^k$, say $q \leq 2^k$, then $0 = c \partial^2^k(xy) = cx \partial^2^k y$ for all $x, y \in R$, and consequently $cI = (0)$. Thus, by the semiprimeness of $R$, $c = 0$, a contradiction. Hence both $p$ and $q$ are greater than $2^k$. For any $x, y \in R$,

$$0 = \partial^m(\partial^{p-2^k-1}xc\partial^{q-2^k}y) = \partial^{2^k-1}\partial^{2^k}(\partial^{p-2^k-1}xc\partial^{q-2^k}y)$$

$$= \partial^{2^k-1}(\partial^{p-1}xc\partial^{q-2^k}y) = \partial^{p-1}xc\partial^{q-2^k}y.$$

If $c \partial^{q-1}y = 0$ for all $y \in R$, then $(p, q - 1) \in S$, contradicting the minimality of $(p, q)$ in $S$. Hence $c_0 = c \partial^{q-1}y_0 \neq 0$ for some $y_0 \in R$. Evidently, $c_0 \in I$, $\partial c_0 = 0$ and moreover, $c_0 \partial^q y = c \partial^{q-1}y_0 \partial^q y = c \partial^2(\partial^{q-1}y_0 y) = 0$ for all $y \in R$.

Thus $(p - 1, q) \in S$, again contradicting the minimality of $(p, q)$ in $S$. Therefore $\partial^{m-1} = 0$.

Now we are in a position to prove our second result.

**THEOREM 2.** Let $R$ be a semiprime ring and $\partial$ a nilpotent derivation of $R$. Then the nilpotency of $\partial$ is either a power of $2$ or an odd number.

**Proof.** Let $I = \{x \in R | 2x = 0\}$. The theorem follows immediately from [1] if $I = (0)$. We thus assume that $I \neq (0)$. Note that $R/I$ is a 2-torsion free semiprime ring and the induced derivation $\bar{\partial}$ of $R/I$ defined by $\bar{\partial}(x + I) = \partial x + I$ is nilpotent. According to [1] the nilpotency of $\bar{\partial}$ is an odd number, say $2n + 1$, i.e., $\partial^{2n+1}R \subseteq I$ but $\partial^{2n}R \not\subseteq I$. On the other hand, $I$ itself is a semiprime ring of characteristic 2 and the restriction of $\partial$ on $I$ is a nilpotent derivation of $I$. By Theorem 1 the nilpotency of $\partial$ on $I$ is a power of 2, say $2^k$. We claim that $\partial^{2k} R \cap I = (0)$. Indeed, for any $a \in I$ and $x \in R$, $0 = \partial^{2k}(xa) = \partial^{2k} xa + x\partial^{2k} a = \partial^{2k} xa$ by noting that $2a = 0$, i.e. $\partial^{2k} R \subseteq I^1$, the left annihilator of $I$. Consequently, $\partial^{2k} R \cap I \subseteq I^1 \cap I$ which is $(0)$ by the semiprimeness of $R$. Thus $\partial^{2k} R \cap I = (0)$.

If $2n + 1 > 2^k$, then $\partial^{2n+1}R \subseteq \partial^{2k} R \cap I$ and hence $\partial^{2n+1}R = (0)$. The nilpotency of $\partial$ is $2n + 1$ as $\partial^{2n}R \neq (0)$.

If $2n + 1 < 2^k$, then $\partial^{2k} R \subseteq \partial^{2n+1}R \subseteq I$. It follows that $\partial^{2k} R = (0)$ and since $(0) \neq \partial^{2k-1} R \subseteq \partial^{2k-1}R$, the nilpotency of $\partial$ is $2^k$. This completes the proof.

We should note that in Theorems 1 and 2, the hypothesis that $R$ is semiprime is essential. It can be seen from the following example.

**EXAMPLE.** Let $R$ be the ring of $6 \times 6$ upper triangular matrices over $GF(2)$ and $\partial$ the inner derivation determined by the matrix $A = E_{12} + E_{23} + E_{34} + E_{45} + E_{56}$, where $E_{ij}$ denotes the unit matrix having 1 at the $(i,j)$-position and zero elsewhere.

One can see easily that $A^6 = 0$, $A^5 \neq 0$ and the nilpotency of $\partial$ is 6 which is neither a power of 2 nor an odd number.

**REFERENCES**


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