A RESULT CONCERNING ADDITIVE FUNCTIONS
IN HERMITIAN BANACH *-ALGEBRAS
AND AN APPLICATION

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Abstract. Let A be a complex hermitian Banach *-algebra with an identity
element e. Suppose there exists an additive function f: A — A such that
f(a) = -a*a(a^-1) holds for all normal invertible elements a £ A. We prove
that in this case f is of the form f(a) = f(ie)k, where a = h + ik. Using this
result we generalize S. Kurepa's extension of Jordan-Neumann characterization
of pre-Hilbert space.

This research has been inspired by the work of S. Kurepa [2, 3] and P. Vrbová
[6]. All algebras and vector spaces in this paper will be over the complex field.
Algebras are assumed to have an identity element, which will be denoted by e. An
algebra A is called a *-algebra if there exists an involution (conjugate-linear anti-
isomorphism of period two) a -+ a* on A. An element h £ A is said to be hermitian
if h* = h, and u £ A is said to be unitary if u*u = uu* = e. An element a £ A
will be called normal if a*a = aa*. It is easy to see that each element a £ A has
a unique decomposition a = h + ik with hermitian h and k. An element a £ A is
normal if and only if h and k commute.

A *-algebra which is also a Banach algebra is called a Banach *-algebra. A
Banach *-algebra is called hermitian if each hermitian element has real spectrum.
Let A be a hermitian Banach *-algebra and let h £ A be a hermitian element. It is
convenient to write h > 0 (h £ 0) if the spectrum of h is positive (nonnegative). The
notation h > k (h £ k) means h - k > 0 (h - k £ 0). The most important hermitian
Banach *-algebras are B*-algebras (i.e. Banach *-algebras in which ||a*a|| = ||a||2
is fulfilled for all a). For basic facts concerning hermitian Banach *-algebras, we
refer the reader to V. Pták's paper [5].

Let X and A be a vector space and an algebra, respectively. Suppose that X is
a left A-module. A left A-module X will be called unitary if ex = x holds for all
x £ X, and will be called irreducible if for each pair x, y £ X, x £ 0, there exists
a £ A such that ax = y.

First we shall consider the following result.

Theorem 1. Let A be a hermitian Banach *-algebra. Suppose there exists an
additive function f : A — A such that f(a) = -a*a(a^-1) holds for all normal
invertible elements a £ A. In this case f(a) = f(ie)k is fulfilled for all a = h + ik.

Remark. If A is the complex number field, then the theorem above reduces to
a result due to P. Vrbová [6].

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vector space, module, additive function, A-bilinear form, A-quadratic form.
For the proof of Theorem 1 we need the lemma below. We omit the proof since it is an easy consequence of Ford's square root lemma [1 or 5, (1.5)].

**Lemma 2.** Let $A$ be a hermitian Banach $^*$-algebra. For each $h > 0$ there corresponds $u > 0$, such that $u^2 = h$. Moreover, $u$ commutes with each element which commutes with $h$.

**Proof of Theorem 1.** Let us first prove that

\[ f(h) = 0 \]

holds for all $h \in A$, $0 < h < e$. Since in this case $e - h^2 > 0$, there exists, by Lemma 2, a hermitian element $k$, such that $k$ commutes with $h$, and $e - h^2 = k^2$, whence it follows that $u = h + ik$ is a unitary element. Therefore, according to the requirements of the theorem, we have

\[ f(h) + f(ik) = f(u) = -u^*u^f(u^{-1}) = -f(h - ik) = -f(h) + f(ik), \]

whence it follows $f(h) = 0$. Let us prove that

\[ f(te) = 0 \]

for each real number $t$. If $0 \leq t < 1$, then (2) follows from (1). It is easy to see that $f(e) = 0$. Therefore (2) holds for all $t \in [0, 1]$. If $t > 1$, we have $0 < t^{-1} < 1$, whence $f(te) = -t^2f(t^{-1}e) = 0$, which proves that (2) holds for all nonnegative real numbers and therefore also for all real numbers. Let us prove that

\[ f(h) = 0 \]

for all hermitian $h \in A$. Therefore, let $h$ be an arbitrary hermitian element, and let us choose a real number $t$ such that $te + h > e$. Then $0 < (te + h)^{-1} < e$. According to (1) we have $f(te + h) = -(te + h)^2f((te + h)^{-1}) = 0$. Hence $f(h) = f(-te)$ and, according to (2), $f(h) = 0$. Now we intend to prove that

\[ f(ih) = hf(ie) \]

holds for all $h \in A$, $0 < h < e$. From $0 < h < e$ it follows that $h - h^2 > 0$. By Lemma 2 there exists a hermitian element $k$, such that $k$ commutes with $h$, and that $h - h^2 = k^2$. The element $a = k + ih$ is normal, since $h$ and $k$ commute. Since $a$ can be expressed in the form $a = h(h^{-1}k + ie)$, it is obvious that $a$ is invertible (recall that $A$ is by assumption hermitian). Therefore using the requirements of the theorem and (3) we obtain

\[ f(ih) = f(k) + f(ih) = f(a) = -a^*af(a^{-1}) = -a^*af((a^*a)^{-1}a^*); \]

\[ = -(h^2 + k^2)f((h^2 + k^2)^{-1}(k - ih)) = -hf(h^{-1}(k - ih)) = -hf(h^{-1}k) + hf(ie) = hf(ie). \]

Let us prove that

\[ f(ite) = tf(ie) \]

holds for each real number $t$. If $0 \leq t \leq 1$, then (5) follows from (4). If $t > 1$, then $0 < t^{-1} < 1$, and we have

\[ f(ite) = -t^2f((ite)^{-1}) = -t^2f(-it^{-1}e) = t^2t^{-1}f(ie) = tf(ie), \]
which proves that (5) holds for all nonnegative real numbers and therefore also for all real numbers. We shall prove that

\[ f(ih) = hf(ie) \]

is fulfilled for each hermitian element \( h \in \mathcal{A} \). Therefore, let \( h \) be an arbitrary hermitian element, and let us choose a real number \( t \) such that \( te + h > e \). Then \( 0 < (te + h)^{-1} < e \). According to (4) we have

\[ f(i(te + h)) = -(te + h)^2 f((i(te + h))^{-1}) = (te + h)^2(te + h)^{-1} f(ie) = (te + h)f(ie). \]

Using the additivity of the function \( f \) and (5), we obtain \( f(ih) = hf(ie) \). From (3), (6) and the fact that each \( a \in \mathcal{A} \) can be expressed in the form \( a = h + ik \), where \( h \) and \( k \) are hermitian, it follows \( f(a) = kf(ie) \), which completes the proof of the theorem.

Let \( X \) and \( \mathcal{A} \) be a complex vector space and a complex \(*\)-algebra, respectively. Suppose that \( X \) is a left \( \mathcal{A} \)-module. A mapping \( B(\cdot, \cdot): X \times X \to \mathcal{A} \) is called an \( \mathcal{A} \)-bilinear form if

1° \( B(a_1 x_1 + a_2 x_2, y) = a_1 B(x_1, y) + a_2 B(x_2, y), \quad x_1, x_2, y \in X, \quad a_1, a_2 \in \mathcal{A}, \)

2° \( B(x, a_1 y_1 + a_2 y_2) = B(x, y_1) a_1^* + B(x, y_2) a_2^*, \quad x, y_1, y_2 \in X, \quad a_1, a_2 \in \mathcal{A}. \)

A mapping \( Q: X \to \mathcal{A} \) is called an \( \mathcal{A} \)-quadratic form if

3° \( Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y \in X, \)

4° \( Q(ax) = a^2 Q(x)a^*, \quad x \in X, \quad a \in \mathcal{A}. \)

Let us consider two examples of \( \mathcal{A} \)-bilinear forms.

**EXAMPLE 1.** Let \( \mathcal{A} \) be a \(*\)-algebra and \( \mathcal{L} \subset \mathcal{A} \) a left ideal. Considering \( \mathcal{L} \) as a left \( \mathcal{A} \)-module, one can introduce an \( \mathcal{A} \)-bilinear form \( B(\cdot, \cdot) \) as follows \( B(x, y) = xy^* \), \( x, y \in \mathcal{L} \).

**EXAMPLE 2.** Let \( X \) be a Hilbert space and let us denote by \( L(X) \) the algebra of all bounded linear operators of \( X \) into itself. Let the involution on \( L(X) \) be the adjoint operation. \( X \) can be considered as a unitary irreducible left \( L(X) \)-module (multiplication by \( A \in L(X) \) is operator action on \( X \)). A simple calculation shows that the mapping \( B(\cdot, \cdot): X \times X \to L(X) \) defined by the relation \( B(x, y)z = (z, y)x \), where \( (\cdot, \cdot) \) denotes the inner product in \( X \), is an \( \mathcal{A} \)-bilinear form.

It is easy to see that each \( \mathcal{A} \)-bilinear form gives rise to the \( \mathcal{A} \)-quadratic form by the relation \( Q(x) = B(x,x) \). It seems natural to ask whether the converse is also true. More precisely, we consider the following

**PROBLEM.** Let \( X \) and \( \mathcal{A} \) be a vector space and a \(*\)-algebra, respectively. Suppose that \( X \) is a left \( \mathcal{A} \)-module, and that there exists an \( \mathcal{A} \)-quadratic form \( Q: X \to \mathcal{A} \) such that \( Q(x) = B(x,x) \) holds for all \( x \in X \)?

It follows from a result of S. Kurepa [3] that the answer to the question above is affirmative if \( \mathcal{A} \) is the complex number field. His result can be formulated as follows.

**THEOREM 3 (S. KUREPA [3]).** Let \( X \) be a vector space over the complex field \( \mathbb{C} \). Suppose there exists a mapping \( Q: X \to \mathbb{C} \) such that \( Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad Q(\lambda x) = |\lambda|^2 Q(x) \) holds for all pairs \( x, y \in X \) and all \( \lambda \in \mathbb{C} \). Under these conditions the mapping \( B(\cdot, \cdot): X \times X \to \mathbb{C} \) defined by

\[ B(x, y) = \frac{1}{4} (Q(x+y) - Q(x-y)) + \frac{i}{4} (Q(x+iy) - Q(x-iy)) \]
is additive in both arguments, and \( B(\lambda x, y) = \lambda B(x, y), B(x, \lambda y) = \lambda B(x, y) \) hold for all pairs \( x, y \in X \) and all \( \lambda \in \mathbb{C} \). For each \( x \in X \) the relation \( Q(x) = B(x, x) \) is fulfilled.

REMARKS. The theorem above can be considered as an extension of the well-known result due to P. Jordan and J. von Neumann which characterizes pre-Hilbert space among all normed spaces. It should be mentioned that P. Vrbová [6] has obtained a simple proof of S. Kurepa’s theorem. Using Theorem 1 and an approach from [6] we prove the result below which can be considered as a generalization of Theorem 3.

THEOREM 4. Let \( X \) be a vector space and \( A \) a commutative hermitian Banach *-algebra. Let \( X \) be a unitary \( A \)-module, and suppose that there exists an \( A \)-quadratic form \( Q : X \rightarrow A \). In this case the mapping \( B(\cdot, \cdot) : X \times X \rightarrow A \) defined by

\[
B(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y)) + \frac{i}{4}(Q(x + iy) - Q(x - iy))
\]

is an \( A \)-bilinear form. For all \( x \in X \) the relation \( Q(x) = B(x, x) \) holds.

PROOF. Let us first prove that the function \( S(\cdot, \cdot) \) defined by the relation \( S(x, y) = Q(x + y) - Q(x - y) \) is additive in both variables. This part of the proof goes through as in the proof of Theorem 3 (see [3] and also [4] for some generalizations), but we shall write it down for the sake of completeness. It is easy to see that \( Q(0) = 0 \) and \( Q(-x) = Q(x) \), \( x \in X \). For arbitrary elements \( x, y, u \in X \) we have

\[
S(x + y, 2u) = Q((x + y) + 2u) - Q((x + y) - 2u)
\]

\[
= Q((x + u) + (y + u)) + Q((x + u) - (y + u)) - Q((x - u) + (y - u)) - Q((x - u) - (y - u)).
\]

Using the relation \( Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \) we obtain

\[
S(x + y, 2u) = 2Q(x + u) + 2Q(y + u) - (2Q(x - u) + 2Q(y - u))
\]

\[
= 2S(x, u) + 2S(y, u).
\]

Hence

\[
S(x + y, 2u) = 2S(x, u) + 2S(y, u).
\]

Putting \( y = 0 \), \( x = z \) we obtain \( S(z, 2u) = 2S(z, u) \). Substituting \( z \) by \( x + y \) and using (7) we finally obtain

\[
2S(x + y, u) = S(x + y, 2u) = 2S(x, u) + 2S(y, u)
\]

which proves that the function \( S(\cdot, \cdot) \) is additive in the first variable. Since \( S(x, y) = S(y, x) \) for all pairs \( x, y \in X \) (this follows from the relation \( Q(-x) = Q(x) \)) it follows that the function \( S(\cdot, \cdot) \) is additive also in the second variable. From the fact that \( S(\cdot, \cdot) \) is additive in both variables, it follows that the same is true for the function \( B(\cdot, \cdot) \) defined by the relation \( B(x, y) = \frac{1}{4}S(x, y) + \frac{i}{4}S(x, iy) \). Therefore, since it is easy to see that \( Q(x) = B(x, x) \) holds for all \( x \in X \), it remains to prove that

\[
B(ax, y) = aB(x, y), \quad B(x, ay) = a^*B(x, y)
\]
is fulfilled for all pairs \( x, y \in X \) and all \( a \in A \). Now we are going to use the condition 

\[ Q(ax) = a^*aQ(x) \]

First of all it follows from the condition above that

\[ S(ax, y) = a^*aS(x, a^{-1}y) \]

holds for all pairs \( x, y \in X \) and all invertible \( a \in A \). Let us prove that \( B(\cdot, \cdot) \) satisfies the relations

\[ B(ix, y) = iB(x, y), \]

\[ B(x, iy) = -iB(x, y) \]

Indeed,

\[ 4B(ix, y) = S(ix, y) + iS(ix, iy) = S(x, -iy) + iS(x, y) \]

\[ = i(S(x, y) - iS(x, -iy)) = i(S(x, y) + iS(x, iy)) = 4iB(x, y) \]

which proves (10). Furthermore,

\[ 4B(x, iy) = S(x, iy) + iS(x, -y) = S(x, iy) - iS(x, y) \]

\[ = -i(S(x, y) + iS(x, iy)) = -4iB(x, y). \]

Now we intend to prove that for the function \( f: A \to A \), defined by the relation

\[ f(a) = B(ax, y) - B(x, ay), \]

where \( x \) and \( y \) are fixed vectors, the requirements of Theorem 1 are fulfilled. Since the additivity of the function above follows from the fact that \( B(\cdot, \cdot) \) is additive in both variables, it remains to show that \( f(a) = -a^*af(a^{-1}) \) holds for all invertible \( a \in A \). We have

\[ 4f(a) = S(ax, y) + iS(ax, iy) - (S(x, ay) + iS(x, iay)) \]

\[ = S(ax, y) + iS(ax, iy) - (S(ay, x) + iS(iay, x)). \]

Using (9) we obtain

\[ 4f(a) = a^*a(S(x, a^{-1}y) + iS(x, ia^{-1}y)) - a^*a(S(y, a^{-1}x) + iS(iy, a^{-1}x)) \]

\[ = 4a^*aB(x, a^{-1}y) - B(a^{-1}x, y)) = -4a^*af(a^{-1}). \]

According to Theorem 1 we have \( f(h + ik) = f(ie)k \) for all hermitian \( h \) and \( k \). In particular, \( f(h) = 0 \) which implies

\[ B(hx, y) = B(x, hy) \]

for all hermitian \( h \in A \) and all pairs \( x, y \in X \). If we put \( a = ih, h \) hermitian, we obtain

\[ B(ihx, y) - B(x, ihy) = f(ih) = hf(ie) = h(B(ix, y) - B(x, iy)). \]

Using (10), (11) and (13) we obtain

\[ B(hx, y) = hB(x, y). \]

Therefore according to (10), (11), (13) and (14) it follows that (8) holds. The proof of the theorem is complete.

**Remark.** It would be interesting to know whether Theorem 4 holds also in the noncommutative case.

We conclude with the following purely algebraic result.
THEOREM 5. Let $X$ be a vector space and $A$ a commutative $\ast$-algebra. Let $X$ be a unitary irreducible $A$-module, and suppose that there exists an $A$-quadratic form $Q: X \to A$. In this case the mapping $B(\cdot, \cdot): X \times X \to A$ defined by

$$B(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y)) + \frac{i}{4}(Q(x + iy) - Q(x - iy))$$

is an $A$-bilinear form. For all $x \in X$ the relation $Q(x) = B(x, x)$ holds.

PROOF. It remains to prove that

$$S(hx, y) = hS(x, y), \quad S(x, hy) = hS(x, y),$$

where $S(x, y)$ stands for $Q(x + y) - Q(x - y)$, holds for all pairs $x, y \in X$ and each hermitian $h \in A$, since the rest of the proof goes through as in the proof of Theorem 4. Therefore, let $x, y \in X$, $h \in A$, $h^* = h$ be arbitrary, and let us prove (15). We may assume that $x \neq 0$, since there is nothing to prove if $x = y = 0$. By the requirements of the theorem there exists $a \in A$ such that $y = ax$. We have

$$S(hx, y) = S(hx, ax) = Q((h + a)x) - Q((h - a)x)$$

$$= (h + a)^*(h + a)Q(x) - (h - a)^*(h - a)Q(x)$$

$$= h((e + a)^*(e + a) - (e - a)^*(e - a))Q(x)$$

$$= h(Q((e + a)x) - Q((e - a)x)) = hS(x, y).$$

Similarly, we obtain that the relation $S(x, hy) = hS(x, y)$ holds. The proof of the theorem is complete.

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