UNIFORM ALGEBRAS AND PROJECTIONS
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ABSTRACT. If $M$ is a closed $A$-submodule of $C(X)$ where $A$ is a uniform algebra on $X$ which contains a separating family of unimodular functions, and if $M$ is a quotient space of some $C(Y)$, then $M$ is an ideal in $C(X)$. If there is an example of a uniform algebra $A$ on some $X$ such that $A \neq C(X)$ but $A$ is complemented in $C(X)$, then there is such an example with $A$ separable.

1. Statements of results. The following problem was considered by Glicksberg [1]: If $A$ is a uniform algebra on $X$ which is (topologically linearly) complemented in $C(X)$, does it follow that $A = C(X)$? He obtained a number of positive results; subsequent work is summarized in Pelczyński's monograph [2].

We shall prove two results. The first reduces to the solution of a special case of Glicksberg's problem provided $M = A$, $Y = X$, and the given linear mapping is a projection (=idempotent linear transformation) on $C(X)$; the case $M = A$ can be readily deduced from Corollary 5.3 of [2]. The second result says that if the answer to Glicksberg's question is negative for some $A$, then it is negative for a separable $A$, a reduction which may prove useful in settling the problem. Recall that a uniform algebra on $X$ is a closed point-separating subalgebra (over $C$) of $C(X)$ which contains the constant functions.

**THEOREM 1.** Let $A$ be a uniform algebra on $X$ which contains enough unimodular functions to separate the points of $X$, and let $M$ be a closed $A$-submodule of $C(X)$. If $M$ is the range of a continuous linear mapping from some $C(Y)$, then $M$ is an ideal in $C(X)$.

**THEOREM 2.** Suppose there is a uniform algebra $A$ on some $X$ such that $A \neq C(X)$ but $A$ is complemented in $C(X)$. Then there is a separable uniform algebra $\hat{A}$ on some $\hat{X}$ such that $\hat{A} \neq C(\hat{X})$ but $\hat{A}$ is complemented in $C(\hat{X})$.

If $K$ is a nonempty closed subset of $X$ then the ideal $I = \{ f \in C(X) : f(x) = 0 \ \forall x \in K \}$ is necessarily complemented in $C(X)$ if $X$ is metrizable. However, if $X = \beta\mathbb{Z}$ the Stone-Čech compactification of the integers $\mathbb{Z}$ and if $K = X \setminus \mathbb{Z}$ then $I$ is not complemented in $C(X)$, that is, $c_0$ is not complemented in $l_\infty$; indeed, $c_0$ is not even a continuous linear image of $l_\infty$. Thus Theorem 1 has no obvious converse in the nonmetrizable case.

In Theorem 2, separability of $\hat{A}$ is equivalent to metrizability of $\hat{X}$. The proof will exhibit $\hat{A}$ as a subalgebra of $A$ and $\hat{X}$ as a quotient space of $X$; thus $\hat{A}$ will be antisymmetric if $A$ is. It will be clear that the proof can be adapted to prove variants of the theorem in which, for example, complementedness is replaced by being a continuous linear image of some $C(Y)$.
2. Proofs. The proof of Theorem 1 is modeled on that of [2, Proposition 4.1].

**Proof of Theorem 1.** Let $K = \bigcap \{ f^{-1}(0) : f \in M \}$, $I = \{ f \in C(X) : f(x) = 0 \ \forall x \in K \}$. We must show that $M = I$. Suppose $M \neq I$. Then there is $\mu \in M$ not supported entirely in $K$. Thus there is $g_0 \in M$ not vanishing identically on the support of $\mu$, so $0 \neq g_0 \mu \in A$. Let $S$ denote the multiplicative semigroup of unimodular functions in $A$. By the Stone-Weierstrass theorem there are $u, v$ in $S$ such that $(g_0 \mu)(u/v) \neq 0$. For $\nu \in M(X) (=\text{regular complex Borel measures on } X)$ let $\nu^* \in M(T)$ ($T$ the unit circle in the complex plane) be $\nu^*(E) = \nu(v^{-1}(E))$, that is, $\int_T f \ d\nu^* = \int_X (f \circ v) \ dv$ for $f \in C(T)$. For each $g \in M$,

$$\int_T z^n d(guv^{-1})^* = \int_X z^n d(guv^{-1} \mu) = \int_X (guv^{-1}) \ d\mu = 0$$

for each positive integer $n$, so by the F. and M. Riesz theorem, $\Phi g = (guv^{-1} \mu)^*$ lies in $H^1 m$ where $m$ is Lebesgue measure on $T$ and $H^1$ is the usual Hardy space. Thus $\Phi$ maps $M$ into the separable dual $H^1 m$ and is absolutely summing [2, Definition 0.2]. If $\Psi$ is a continuous linear mapping of $C(Y)$ onto $M$ then $\Phi \circ \Psi$ is absolutely summing, so compact [2, Theorem 0.5]; thus by the open mapping theorem $\Phi$ is compact, which in turn implies that $a \to (a \circ v_0)g_0$ is compact from the disc algebra to $H^1 m$. This, however, is false: if $0 \neq h \in L^1(m)$ (the role of $h$ being played above by $g_0$ where $\int_T d(g_0) = \int_X g_0 v^{-1} \ d\mu = 0$) choose integers $N_1, N_2$ so that $\int z^{N_1} h \ dm = 2\varepsilon > 0$ and (using the Riemann-Lebesgue lemma) $\int z^n h \ dm < \varepsilon$ whenever $|n| > N_2 > |N_1|$; if $n_1, n_2$ are distinct (positive) integers then

$$\|z^{2n_1}h_m - z^{2n_2}h_m\|_{M(T)} = \|z^{N_1}h_m - z^{2n_2}h_m\|_{M(T)} \geq \left| \int z^{N_1} h \ dm \right| - \left| \int z^{N_1+2(n_2-n_1)}h \ dm \right| > 2\varepsilon - \varepsilon = \varepsilon,$$

contradicting the alleged compactness.

**Proof of Theorem 2.** Let $P$ be a continuous projection on $C(X)$ with range $A$. We build increasing sequences $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ and $B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$ of separable closed subalgebras of $C(X)$ with $B_n$ selfadjoint, $A_n \subset B_n \cap A$, and $P(B_n) \subset A_{n+1}$ as follows. Choose $h \in A$ such that $h \notin A$. Let $A_0$ be the smallest closed subalgebra of $C(X)$ that contains $h$ and 1. Then successively let $B_n$ be the closed selfadjoint subalgebra of $C(X)$ generated by $A_n$, and let $A_{n+1}$ be the closed subalgebra of $C(X)$ generated by $P(B_n)$. Let $A'$ and $B'$ denote the respective closures of $\bigcup A_n$ and $\bigcup B_n$. These are separable closed subalgebras of $C(X)$, $B'$ is selfadjoint while $A'$ is not ($h \in A' \subset A$), $A' \subset B'$ and $P(B') = A'$. If $\bar{X}$ is obtained from $X$ by collapsing each common set of constancy for $A'$ (equivalently, for $B'$) to a point, then $B'$ and $A'$ become the required $C(\bar{X})$ and $A$.

**References**


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