MEROMORPHIC MAPS IN THE NEVANLINNA CLASS
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ABSTRACT. A meromorphic mapping from a relatively compact domain \( D \) in a complex manifold to complex projective space is shown to be in the Nevanlinna class provided the mapping omits a set of hyperplanes of positive capacity. As a consequence, such mappings have admissible limits almost everywhere on \( \partial D \).

In this note we give a sufficient condition that a meromorphic mapping of a relatively compact domain in a complex manifold be in the Nevanlinna class. As a consequence we obtain new results on the existence of boundary values of meromorphic mappings. A well-known result of Stein [5] says that a holomorphic function in the Nevanlinna class for a smooth bounded domain in \( \mathbb{C}^n \) has admissible limit almost everywhere on the boundary of the domain. In a recent paper Lempert [1] extended this result to meromorphic functions. The second author showed the existence of admissible limits almost everywhere for meromorphic mappings in the Nevanlinna class [4].

For holomorphic mappings \( F: B^n \rightarrow \mathbb{C}^n \) of the ball, a sufficient condition for \( F \) to be in the Nevanlinna class may be given in terms of capacity [3]. As we shall see below this result extends to meromorphic mappings of arbitrary relatively compact domains with smooth boundary in a complex manifold. One key point in this discussion is the fact that it is not in general possible to write a meromorphic mapping in the Nevanlinna class with given components as quotients of holomorphic functions in the Nevanlinna class in dimensions greater than one.

Let \( M \) be a connected complex manifold of dimension \( m > 1 \) and \( \chi \) a smooth strictly positive form of bidegree \( (m - 1, m - 1) \) on \( M \). Suppose \( D \subset M \) is open, relatively compact, and with smooth boundary. We suppose \( \psi \) is a defining function for \( D \) so

(i) \( D = \{ x \in M : \psi(x) < 0 \} \),
(ii) \( \partial D = \{ x \in M : \psi(x) = 0 \} \),
(iii) \( \psi \) is smooth up to the boundary and \( d\psi \neq 0 \) on \( \partial D \).

Let

\[ D(r) = \{ x \in M : \psi(x) < r \} \quad \text{for} \ r < 0. \]

If

\[ X = [X_0 : \cdots : X_n] \in \mathbb{PC}, \quad A = [A_0 : \cdots : A_n] \in \mathbb{P}^n\mathbb{C}^*, \]

let

\[ \|A : X\| = |A_0X_0 + A_1X_1 + \cdots + A_nX_n| \setminus |X| |A|. \]

For \( A \in \mathbb{P}^n\mathbb{C}^* \), let \( H^A \) denote the hyperplane defined by

\[ H^A = \{ Z \in \mathbb{P}^n\mathbb{C} : \|Z : A\| = 0 \}. \]
Let $F: \mathcal{D} \to \mathbb{P}^n\mathbb{C}$ be a meromorphic mapping. The value distribution functions associated with $F$ are given below.

$$T(r) = \int_{\mathcal{D}(r)} (r - \psi)F^*\Omega \wedge \chi \quad \text{for } r_0 \leq r < 0, \ r_0 \text{ fixed}.$$  

The $(1,1)$ from $\Omega$ is the Kähler form of the Fubini-Study metric on $\mathbb{P}^n\mathbb{C}$.

$$m(r, A) = \int_{\partial \mathcal{D}(r)} \log \frac{1}{\|F : A\|} (2d^{\ast} \psi \wedge \chi)$$

$$= \int_{\partial \mathcal{D}(r)} F^* \left( \log \frac{1}{\|Z : A\|} \right) \cdot (2d^{\ast} \psi \wedge \chi),$$

$$d^c = \frac{\sqrt{-1}}{2\pi} (\bar{\partial} - \partial).$$

Given $A \in \mathbb{P}^n\mathbb{C}^*$, let $\nu^A_F$ denote the $A$-divisor of $F$ and $S_A$ the support of $\nu^A_F$. The integrated counting function is then defined by

$$N(r, A) = \int_{\mathcal{D}(r) \cap S_A} (r - \psi)\nu^A_F \chi, \quad r_0 \leq r < 0.$$  

The deficit is given by

$$D(r, A) = \int_{\mathcal{D}(r) \setminus \mathcal{D}(r_0)} \log \frac{1}{\|F : A\|} \ d(2d^{\ast} \psi \wedge \chi).$$

The First Main Theorem (FMT) then states [6 or 7]

$$T(r) + D(r, A) = N(r, A) + m(r, A) - m(r_0, A).$$

The functions appearing in the FMT above depend on the choices of $\psi$ and $\chi$; however, the boundedness of $T(r)$ is independent of the choices. The following definition is therefore independent of the functions $\psi$ and $\chi$.

**Definition.** $F: \mathcal{D} \to \mathbb{P}^n\mathbb{C}$, a meromorphic mapping, is said to be in the Nevanlinna class, $N(\mathcal{D})$, provided $T(r)$ is bounded.

We are interested in obtaining a sufficient condition that a meromorphic mapping be in $N(\mathcal{D})$ in terms of value distribution properties of $F$. We use the following Blaschke type condition.

**Definition.** $F: \mathcal{D} \to \mathbb{P}^n\mathbb{C}$, a meromorphic mapping, satisfies the Blaschke condition for $A$ provided $N(r, A)$ is bounded as a function of $r$.

The value distribution property of $F$ we use involves the capacity of a set of hyperplanes. Making the identification of hyperplanes in $\mathbb{P}^n\mathbb{C}$ with points in $\mathbb{P}^n\mathbb{C}^*$ we define the capacity of a Borel measurable set $E$ as:

**Definition.** If $E \subset \mathbb{P}^n\mathbb{C}^*$ let

$$V(E) = \inf_{\mu \in \mathcal{P}(E)} \sup_{Z \in \mathbb{P}^n\mathbb{C}} \int_{A \in E} \log \frac{1}{\|Z : A\|} \ d\mu(A), \quad C(E) = 1/V(E),$$

where $\mathcal{P}(E)$ denotes the set of probability measures supported on $E$. For properties of this capacity see [2].

We can now give a sufficient condition that a meromorphic mapping be in the Nevanlinna class.
**THEOREM.** With $D \subset M$, $\psi$ and $\chi$ as above, let $F: D \to \mathbf{P}^n \mathbb{C}$ be a meromorphic mapping. Suppose $F$ satisfies the Blaschke condition for all $A \in E$ with $C(E) > 0$. Then $F \in N(D)$.

**PROOF.** Since $N(r, A) < \infty$ for all $A \in E$ and $C(E) > 0$, there exists a set $E_0 \subset E$ with $N(r, A) \leq M_0$ for all $A \in E_0$ and $C(E_0) > 0$. Let $\mu_0$ be an equilibrium measure associated with $E_0$. Integrate the FMT with respect to $\mu_0$ to get

$$T(r) = \int_{E_0} N(r, A) d\mu_0(A) + \int_{E_0} m(r, A) d\mu_0(A)$$

$$- \int_{E_0} m(r_0, A) d\mu_0(A) - \int_{E_0} D(r, A) d\mu_0(A).$$

Since

$$\int N(r, A) d\mu_0(A) \leq M_0,$$

we need only show that the other integrals are finite. We show that the last integral is finite; the others follow in a similar manner.

$$\int_{E_0} D(r, A) d\mu_0(A) = \int_{E_0} \int_{D(r) \setminus D(\bar{r})} F^* \left( \log \frac{1}{\|Z : A\|} \right) d(2d^c \psi \wedge \chi) d\mu_0(A)$$

$$= \int_{D(r) \setminus D(\bar{r})} \left[ F^* \int_{E_0} \log \frac{1}{\|Z : A\|} d\mu_0(A) \right] d(2d^c \psi \wedge \chi).$$

Since $C(E_0) > 0$, we have the estimate

$$\int_{E_0} \log \frac{1}{\|Z : A\|} d\mu_0(A) \leq V(E_0) = 1/C(E_0).$$

Hence

$$\int_{E_0} D(r, A) d\mu_0(A) \leq \int_{D(r) \setminus D(\bar{r})} 1/C(E_0) \cdot d(2d^c \psi \wedge \chi) \leq 1/C(E_0) \cdot k,$$

where $k$ is independent of $r$ (but depends of course on the choices of $\psi$ and $\chi$). \(\square\)

In [4], the second author shows that meromorphic mappings in the Nevanlinna class have admissible limits almost everywhere $(2m - 1$ Hausdorff measure $\mathcal{H}_{2m-1}$) on $\partial D$. We therefore have immediately the following result.

**COROLLARY.** Let $D \subset M$ be as above. Suppose $F: D \to \mathbf{P}^n \mathbb{C}$ is a meromorphic mapping such that $F$ satisfies the Blaschke condition for all $A \in E \subset \mathbf{P}^n \mathbb{C}^*$ with $C(E) > 0$. Then $F$ has admissible limits almost everywhere $\mathcal{H}_{2m-1}$ on $\partial D$.

**REMARKS.** It is possible to localize the above results in the following sense. If $D \subset M$ is any domain with smooth boundary, choose $\rho \in D$. Then $F: D \to \mathbf{P}^n \mathbb{C}$ meromorphic is in the local Nevanlinna class at $\rho$, $N_{\text{loc}}(\rho, D)$, if there exists a coordinate neighborhood $U \subset M$ about $\rho$ such that $F \in N(D \cap U)$. One may then formulate local versions of the above results. See Stein [5] for a comparison.

**REFERENCES**


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