Let $\mu$ denote a finite positive Borel measure with compact support in the complex plane. Let $P^2(\mu)$ denote the closure in $L^2(\mu)$ of the polynomials in $z$. A question of interest is to determine when $P^2(\mu) = L^2(\mu)$. If $\mu$ is supported on the boundary of the unit circle, $\partial D$, such a characterization has been given by a classical result of Szegö [5]: either $P^2(\mu) = L^2(\mu)$ or else $P^2(\mu)$ has a bounded point evaluation. $P^2(\mu)$ has a bounded point evaluation (or b.p.e.) at $w$ in the complex plane $\mathbb{C}$ whenever there exists a constant $C$ with $0 < C < \infty$ and $|p(w)| < C||p||_2(\mu)$ for all $p \in P^2(\mu)$. For measures $\gamma$ absolutely continuous with respect to area Lebesgue measure $m$, a result analogous to Szegö's theorem has been discovered by Brennan [1–4] and Hruschev (see [4]), with the mild hypothesis that $d\gamma/dm$ belong to $L(\log^+ L)^2(m)$. In this note we show that if $P^2(\mu)$ has no b.p.e.'s, then $P^2(\mu)$ and $L^2(\mu)$ cannot differ by "too" much. As a consequence, to show that $P^2(\mu) = L^2(\mu)$ in the absence of bounded point evaluations, one need only show that, for example, $\sqrt{z} - \lambda \in P^2(\mu)$ for complex $\lambda$'s.

Denote the support of $\mu$ by $K$. Let $g$ be a continuously differentiable function on $\mathbb{C}$ with $\bar{\partial}g$ nonvanishing on $K$, where $\bar{\partial}$ denotes the operator $1/2(\partial x - i\partial y)$. Let $\{g_i\}_{i \in I} \subset L^\infty(\mu)$. By $\operatorname{sp}\{g_iP^2(\mu): i \in I\}$ we mean the $\{\sum^K_{j=1}g_{ij}p_j: i_j \in I \text{ and } p_j \in P^2(\mu) \text{ for } j = 1, 2, \ldots, K\}$.

**Theorem 1.** Suppose $P^2(\mu)$ has no b.p.e.'s. Then $\operatorname{sp}\{P^2(\mu) + gP^2(\mu)\} = L^2(\mu)$.

**Proof.** Let $f \in L^2(\mu)$ with $f \perp [P^2(\mu) + gP^2(\mu)]$. Then for all $\lambda \in \mathbb{C}$ and any polynomial $p$

$$0 = \int_K \frac{p(z) - p(\lambda)}{z - \lambda} (g(z) - g(\lambda)) f(z) d\mu(z).$$

Received by the editors February 18, 1983 and, in revised form, August 25, 1983.

1980 Mathematics Subject Classification. Primary 47B20, 46J99; Secondary 30A82.

Key words and phrases. $P^2(\mu)$, bounded point evaluation.

1Partially supported by a grant from the Research Grants Committee of the University of Alabama.

©1984 American Mathematical Society

0002-9939/84 $1.00 + .25 per page

421
so

\begin{equation}
(1) \quad p(\lambda) \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) = \int_K p(z) \left( \frac{g(z) - g(\lambda)}{z - \lambda} \right) \overline{f(z)} \, d\mu(z).
\end{equation}

Without loss of generality assume that \( g \in C_c^1(\mathbb{C}) \). Thus for any

\[ \frac{g(z) - g(\lambda)}{z - \lambda} f(z) \in L^2(\mu). \]

Suppose that for some \( \lambda \) in \( \mathbb{C} \)

\[ \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) \neq 0. \]

Then from (1) \( P^2(\mu) \) has a b.p.e. at \( \lambda \), contrary to hypothesis. Hence

\begin{equation}
(2) \quad 0 = \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) \quad \text{for all } \lambda \in \mathbb{C}.
\end{equation}

But by Lemma 3 in [6], for any \( h \in C_c^2(\mathbb{C}) \) with \( \lambda = 0 \) in a neighborhood of the zero set of \( \delta g \) we have

\begin{equation}
(3) \quad h(w) = \frac{1}{\pi} \int_C \overline{\delta s(z)} \frac{g(z) - g(w)}{z - w} \, dm(z)
\end{equation}

for all \( w \) in \( \mathbb{C} \) and some \( s \in C_c^1(\mathbb{C}) \). Since \( \delta g \) does not vanish on \( K \) such \( h \)'s are dense in \( L^2(\mu) \). Combining (2) and (3) with Fubini's theorem gives \( f = 0 \) in \( L^2(\mu) \). \( \square \)

Note that it is easy to see that

\[ P^2(\mu) \oplus g P^2(\mu) = L^2(\mu) \]

can happen only in trivial cases.

If it could be shown that \( g \) itself is in \( P^2(\mu) \) when \( P^2(\mu) \) has no b.p.e.'s, then \( P^2(\mu) = L^2(\mu) \) and the main problem is solved. This direct approach seems unlikely since the \( g \)'s for which the theorem holds are far from analytic on \( K \) and thus difficult to place in \( P^2(\mu) \). Perhaps the following version might be more useful. Here \( g \) is replaced by a collection of functions, but each function is analytic except on negligible sets with respect to \( \mu \).

For \( z = re^{i\theta} \) with \( r > 0 \) and \( 0 \leq \theta < 2\pi \), let \( \sqrt{z} = r^{1/2}e^{i\theta/2} \). Then for \( \lambda \) in \( \mathbb{C} \), \( \sqrt{z - \lambda} \) is analytic in \( z \) on \( \mathbb{C} - \{\lambda + t : t \geq 0\} \). It is easy to check that \( (z, \lambda) \mapsto \sqrt{z - \lambda} \) is Borel measurable from \( \mathbb{C} \times \mathbb{C} \to \mathbb{C} \). We have the following theorem.

**THEOREM 2.** Suppose that \( P^2(\mu) \) has no b.p.e.'s. Then

\[ \text{sp}\{P^2(\mu) + \sqrt{z - \lambda} \ P^2(\mu) : \lambda \in \mathbb{C}\} = L^2(\mu). \]

**PROOF.** Suppose that \( f \in L^2(\mu) \) and \( f \perp \{P^2(\mu) + \sqrt{z - \lambda} \ P^2(\mu)\} \) for \( m\)-a.e. \( \lambda \) in \( \hat{K} \), the union of \( K \) and all the "holes" in \( K \). For \( p \) a polynomial

\begin{equation}
(4) \quad 0 = \int_K \frac{p(z) - p(\lambda)}{z - \lambda} \sqrt{z - \lambda} \ f(z) \, d\mu(z) \quad \lambda \text{ a.e.-}m \text{ in } \hat{K}.
\end{equation}

As before we claim that \( P^2(\mu) \) has a b.p.e. unless \( m_2 \)-a.e. \( \lambda \in \hat{K} \) satisfies

\begin{equation}
(5) \quad 0 = \int_K \frac{\sqrt{z - \lambda}}{z - \lambda} \ f(z) \, d\mu(z) = \mu_0(\lambda).
\end{equation}
This follows from (4) since
\[ p(\lambda)\mu_0(\lambda) = \int_K p(z)\left[\frac{\sqrt{z - \lambda}}{z - \lambda} f(z)\right] d\mu(z), \]
and \((\sqrt{z - \lambda}/(z - \lambda))f(z) \in L^2(\mu)\) for \(m\)-a.e. \(\lambda\) in \(\hat{K}\). The last fact holds by Fubini's theorem, since
\[ \int_L \int_K \left|\frac{\sqrt{z - \lambda}}{z - \lambda} f(z)\right|^2 d\mu(z) d\lambda(\lambda) \leq C_L\|f\|_{L^2(\mu)}, \]
where \(L\) is a disc containing \(K\) and \(C_L\) is a constant.

We show that (5) implies that \(f = 0\) in \(L^2(\mu)\). Let \(\phi \in C_c^\infty(L)\). Then
\[ 0 = \int_L \frac{\partial \phi(\lambda)}{\partial \lambda} \left(\int_K \frac{1}{\sqrt{z - \lambda}} f(z) \right) d\mu(z) d\lambda(\lambda) \]
\[ = \int_K f(z) \left(\int_L \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda)\right) d\mu(z). \]
Let \(L_\epsilon\) denote \(L\) with an \(\epsilon\)-strip, \(S_\epsilon\), about the ray \(t + i \text{Im} Z, t \leq \text{Re} Z\), and the disc \(\Delta_\epsilon(z)\) removed. Then
\[ \lim_{\epsilon \to 0} \int_{L_\epsilon} \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda) = \int_L \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda). \]

On the other hand, by Green's Theorem
\[ \int_{L_\epsilon} \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda) = \frac{1}{2i} \left[ \int_{\partial L_\epsilon \cap \partial L_\epsilon} \frac{\phi(\lambda)}{\sqrt{z - \lambda}} d\lambda - \int_{-\pi/2}^{\pi/2} \phi(z + \epsilon e^{i\theta}) e^{i\epsilon} \frac{d \theta}{\sqrt{\epsilon e^{i\theta}}} \right. \]
\[ \left. - \int_{-\infty}^{0} \frac{\phi(t - i\epsilon + z)}{\sqrt{-t + i\epsilon}} dt + \int_{-\infty}^{0} \frac{\phi(t + i\epsilon + z)}{\sqrt{-t - i\epsilon}} dt \right]. \]
The first integral is 0, since \(\phi \in C_c^\infty(L)\). As \(\epsilon \to 0\) the second integral converges to 0, while the square roots in the third and fourth integrands converge to \(\sqrt{-t}\) and \(-\sqrt{-t}\), respectively. Thus
\[ \int_L \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda) = i \int_{-\infty}^{0} \frac{\phi(t + z)}{\sqrt{-t}} dt. \]
Let \(\phi(x, y) = \psi(x)\alpha(y)\), where \(\psi\) and \(\alpha\) are in \(C_c^\infty(R)\). Then
\[ \int_L \frac{\partial \phi(\lambda)}{\partial \lambda} d\lambda(\lambda) = i\alpha(y) \int_{0}^{\infty} \frac{\psi(x - t)}{\sqrt{t}} dt. \]
Fix \(A > \max\{|\text{Re} z|: z \in L\}\). Notice that
\[ \int_{0}^{\infty} (x - t)^n X(x; -A < x < A) \frac{dt}{\sqrt{t}} = \int_{0}^{x + A} (x - t)^n \frac{dt}{\sqrt{t}} \quad \text{for } |x| \leq A. \]
This last expression is \(\sqrt{x + A}\) times a polynomial in \(x\) of degree \(n\), \(p_n(x)\). (The leading coefficient of \(p_n(x)\) is \(\int_0^1 u^{-1/2}(1 - u)^n du > 0\).) Choose \(\psi\)'s to approximate \(t^n X(x; -A, A)\) pointwise boundedly. Combining (6), (7), and (8), we see that \(f \perp \alpha(y)\sqrt{x + A} p_n(x)\) (since the support of \(\mu\) is contained in \(L\)). Thus by the Stone-Weierstrass theorem \(\sqrt{x + A} f = 0\) in \(L^2(\mu)\), so \(f = 0\) in \(L^2(\mu)\).
It is not difficult to replace the branch chosen for Theorem 2 by a more complicated one; say a Jordan arc which is piecewise smooth and rectifiable in $L$. In fact the choice of branch may depend on $\lambda$ if the perturbation is smooth. Also $\sqrt{z - \lambda}$ may be replaced by $\sqrt{\frac{1}{\alpha}z - \lambda} \alpha > 0$, or, for example, by $(z - \lambda)\log(z - \lambda)$. Since the total variation of $\mu$ is finite, the $\mu$ measure of horizontal lines is zero, except for at most a countable set. Thus the $\sqrt{z - \lambda}$'s needed for Theorem 2 may be restricted so that $\sqrt{z - \lambda}$ is analytic $\mu$-a.e.

We prove a similar theorem where the branches of $\sqrt{z}$ vary, but where the base of $\sqrt{z}$ is fixed. For each $a \in [0, 2\pi)$ we define a function $f_a(z) = r^{1/2}e^{i\theta}/2$, where $z = re^{i\theta}$ is chosen so that $a < \theta < 2\pi + a$. For this theorem no assumption concerning b.p.e.'s is needed.

**Theorem 3.** \( sp\{ P^2(\mu) + f_a(z)P^2(\mu); \alpha \in [0, 2\pi) \} = L^2(\mu) \).

**Proof.** Let $0 < \alpha < \beta < 2\pi$. Then

\[
\frac{f_\beta(z) - f_\alpha(z)}{2} = \begin{cases} 
0 & \text{if arg } z \in [\beta, 2\pi + \alpha) \mod 2\pi, \\
-f_\alpha(z) & \text{if arg } z \in [\alpha, \beta) \mod 2\pi.
\end{cases}
\]

Denote by $\mathcal{L}$ the closed linear span of \{ $P^2(\mu) + f_a(z)P^2(\mu); \alpha \in [0, 2\pi)$ \}. From (9) $f_0(z)X(z)z^{\alpha} \in \mathcal{L}$ for every polynomial $p$ and $0 < \alpha < \beta < 2\pi$. Thus by approximation $f_0(z)h(\arg z)z^n \in \mathcal{L}$ for every nonnegative integer $n$ and every continuous function $h$ on $[0, 2\pi]$ with $h(0) = h(2\pi)$. If $m$ is a nonnegative integer take $h(\arg z) = z^m/|z|^{m+n}$; if $m$ is a negative integer let $h(\arg z) = z^{-m}/|z|^{m+n}$. In either case

\[
f_0(z)h(\arg z)z^n = f_0(re^{i\theta})r^ne^{im\theta} \in \mathcal{L},
\]

where $z = re^{i\theta}$. By the Stone-Weierstrass theorem $sp\{ r^ne^{im\theta}; n \in N, m \in Z \}$ is dense in $C(K)$. Thus if $k \in L^2(\mu)$ satisfies $k \perp \mathcal{L}$, then $f_0(z)k(z) \, d\mu$ is the zero measure. Hence $kd\mu = c\delta_0$. But $1 \in \mathcal{L}$ so $k = 0$ in $L^2(\mu)$. \( \square \)

It is clear that only a dense subset of $a$'s in $[0, 2\pi)$ is needed for Theorem 3. Again a similar argument holds for $z\log_a z$, where $\log_a(z) = \ln(z) + i\arg z$ and $\alpha < \arg z < \alpha + 2\pi$. Also a smooth one parameter family of nonintersecting smooth Jordan arcs emanating from a base point to $\infty$ can replace the radial lines.

**Note.** (a) The case $g(z) = \bar{z}$ of Theorem 1 was independently discovered and orally communicated to us by J. Thomson and R. Olin.

(b) When $K$ is simply connected with empty interior, arguments involving b.p.e.'s and a result analogous to Theorem 2 lead to a new proof of a theorem of Lavrentieff on the uniform approximation of continuous functions on $K$ by polynomials.

The authors wish to thank the reviewer for several valuable comments.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA (TUSCALOOSA), UNIVERSITY, ALABAMA 35486