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$P^2(\mu)$ AND BOUNDED POINT EVALUATIONS

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ABSTRACT. It is shown that if $g \in C_c^1(\mathbf{C})$ with $\bar{\partial} g$ nonvanishing on the support of μ and if $P^2(\mu)$ has no bounded point evaluations, then $\operatorname{sp}\{P^2(\mu) + gP^2(\mu)\}^- = L^2(\mu)$. Similar theorems stating that in the absence of bounded point evaluations $P^2(\mu)$ is "almost" $L^2(\mu)$ are derived. As a consequence, to show that $P^2(\mu) = L^2(\mu)$ in the absence of bounded point evaluations, one need only show that, for example, $\sqrt{z-\lambda} \in P^2(\mu)$ for complex λ's.

Let μ denote a finite positive Borel measure with compact support in the complex plane. Let $P^2(\mu)$ denote the closure in $L^2(\mu)$ of the polynomials in z. A question of interest is to determine when $P^2(\mu) = L^2(\mu)$. If μ is supported on the boundary of the unit circle, ∂D, such a characterization has been given by a classical result of Szegö [5]: either $P^2(\mu) = L^2(\mu)$ or else $P^2(\mu)$ has a bounded point evaluation. $P^2(\mu)$ has a bounded point evaluation (or b.p.e.) at w in the complex plane $\mathbf{C}$ whenever there exists a constant C with $0 < C < \infty$ and $|p(w)| \leqslant C\|p\|_{2,\mu}$ for all $p \in P^2(\mu)$. For measures γ absolutely continuous with respect to area Lebesgue measure m, a result analogous to Szegö's theorem has been discovered by Brennan [1–4] and Hruschev (see [4]), with the mild hypothesis that $d\gamma/dm$ belong to $L(\operatorname{Log}^+ L)^2(m)$. In this note we show that if $P^2(\mu)$ has no b.p.e.'s, then $P^2(\mu)$ and $L^2(\mu)$ cannot differ by "too" much. As a consequence, to show that $P^2(\mu) = L^2(\mu)$ in the absence of bounded point evaluations, one need only show that, for example, $\sqrt{z-\lambda} \in P^2(\mu)$ for complex λ's.

Denote the support of μ by K. Let g be a continuously differentiable function on $\mathbf{C}$ with $\bar{\partial} g$ nonvanishing on K, where $\bar{\partial}$ denotes the operator $1/2(\partial x + i\partial y)$. Let $\{g_i\}_{i\in I} \subset L^\infty(\mu)$. By $\operatorname{sp}\{g_i P^2(\mu)\colon i \in I\}$ we mean the $\{\sum_{j=1}^K g_{i_j} p_j\colon i_j \in I$ and $p_j \in P^2(\mu)$ for $j = 1, 2, \ldots, K\}$.

THEOREM 1. *Suppose $P^2(\mu)$ has no b.p.e.'s. Then* $\operatorname{sp}\{P^2(\mu) + gP^2(\mu)\}^- = L^2(\mu)$.

PROOF. Let $f \in L^2(\mu)$ with $f \perp [P^2(\mu) + gP^2(\mu)]$. Then for all $\lambda \in \mathbf{C}$ and any polynomial p

$$0 = \int_K \frac{p(z) - p(\lambda)}{z - \lambda}(g(z) - g(\lambda))\overline{f(z)}\, d\mu(z).$$

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So

$$(1)\quad p(\lambda)\int_K \frac{g(z)-g(\lambda)}{z-\lambda}\overline{f(z)}\,d\mu(z)=\int_K p(z)\left(\frac{g(z)-g(\lambda)}{z-\lambda}\right)\overline{f(z)}\,d\mu(z).$$

Without loss of generality assume that $g \in C_c^1(\mathbf{C})$. Thus for any

$$\frac{g(z)-g(\lambda)}{z-\lambda}f(z)\in L^2(\mu).$$

Suppose that for some λ in $\mathbf{C}$

$$\int_K \frac{g(z)-g(\lambda)}{z-\lambda}\overline{f(z)}\,d\mu(z)\neq 0.$$

Then from (1) $P^2(\mu)$ has a b.p.e. at λ, contrary to hypothesis. Hence

$$(2)\qquad 0=\int_K \frac{g(z)-g(\lambda)}{z-\lambda}\overline{f(z)}\,d\mu(z)\quad \text{for all } \lambda \text{ in } \mathbf{C}.$$

But by Lemma 3 in [**6**], for any $h \in C_c^2(\mathbf{C})$ with $h \equiv 0$ in a neighborhood of the zero set of $\bar\partial g$ we have

$$(3)\qquad h(w)=\frac{1}{\pi}\int_{\mathbf{C}} \bar\partial s(z)\frac{g(z)-g(w)}{z-w}\,dm(z)$$

for all w in $\mathbf{C}$ and some $s \in C_c^1(\mathbf{C})$. Since $\bar\partial g$ does not vanish on K such h's are dense in $L^2(\mu)$. Combining (2) and (3) with Fubini's theorem gives $f = 0$ in $L^2(\mu)$. □

Note that it is easy to see that $P^2(\mu)\oplus gP^2(\mu)=L^2(\mu)$ can happen only in trivial cases.

If it could be shown that g itself is in $P^2(\mu)$ when $P^2(\mu)$ has no b.p.e.'s, then $P^2(\mu)=L^2(\mu)$ and the main problem is solved. This direct approach seems unlikely since the g's for which the theorem holds are far from analytic on K and thus difficult to place in $P^2(\mu)$. Perhaps the following version might be more useful. Here g is replaced by a collection of functions, but each function is analytic except on negligible sets with respect to μ.

For $z = re^{i\theta}$ with $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$, let $\sqrt{z} = r^{1/2}e^{i\theta/2}$. Then for λ in $\mathbf{C}$, $\sqrt{z-\lambda}$ is analytic in z on $\mathbf{C}-\{\lambda+t\colon t\geqslant 0\}$. It is easy to check that $(z,\lambda)\mapsto\sqrt{z-\lambda}$ is Borel measurable from $\mathbf{C}\times\mathbf{C}\to\mathbf{C}$. We have the following theorem.

THEOREM 2. *Suppose that $P^2(\mu)$ has no b.p.e.'s. Then*

$$\operatorname{sp}\{P^2(\mu)+\sqrt{z-\lambda}\,P^2(\mu)\colon \lambda\in\mathbf{C}\}^{-}=L^2(\mu).$$

PROOF. Suppose that $f\in L^2(\mu)$ and $f\perp[P^2(\mu)+\sqrt{z-\lambda}\,P^2(\mu)]$ for m-a.e. λ in $\hat K$, the union of K and all the "holes" in K. For p a polynomial

$$(4)\qquad 0=\int_K \frac{p(z)-p(\lambda)}{z-\lambda}\sqrt{z-\lambda}\,\overline{f(z)}\,d\mu(z)\quad \lambda \text{ a.e.-}m \text{ in } \hat K.$$

As before we claim that $P^2(\mu)$ has a b.p.e. unless m_2-a.e. $\lambda\in\hat K$ satisfies

$$(5)\qquad 0=\int_K \frac{\sqrt{z-\lambda}}{z-\lambda}\overline{f(z)}\,d\mu(z)\equiv\mu_0(\lambda).$$

This follows from (4) since

$$p(\lambda)\mu_0(\lambda) = \int_K p(z)\left[\frac{\sqrt{z-\lambda}}{z-\lambda}\overline{f(z)}\right] d\mu(z),$$

and $(\sqrt{z-\lambda}/(z-\lambda))\overline{f(z)} \in L^2(\mu)$ for m-a.e. λ in $\hat{K}$. The last fact holds by Fubini's theorem, since

$$\int_L\int_K \left|\frac{\sqrt{z-\lambda}}{z-\lambda} f(z)\right|^2 d\mu(z)\, dm(\lambda) \leqslant C_L\|f\|^2_{2,\mu},$$

where L is a disc containing K and C_L is a constant.

We show that (5) implies that $f = 0$ in $L^2(\mu)$. Let $\phi \in C_c^\infty(L)$. Then

$$\begin{aligned}(6)\qquad 0 &= \int_L \bar{\partial}\phi(\lambda)\int_K \frac{1}{\sqrt{z-\lambda}}\overline{f(z)}\, d\mu(z)\, dm(\lambda)\\ &= \int_K \overline{f(z)}\left[\int_L \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda)\right] d\mu(z).\end{aligned}$$

Let L_ε denote L with an ε-strip, S_ε, about the ray $t + i\operatorname{Im} z$, $t \leqslant \operatorname{Re} z$, and the disc $\Delta_\varepsilon(z)$ removed. Then

$$\operatorname*{Lim}_{\varepsilon\downarrow 0}\int_{L_\varepsilon} \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda) = \int_L \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda).$$

On the other hand, by Green's Theorem

$$\begin{aligned}\int_{L_\varepsilon} \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda) = \frac{1}{2i}\Bigg[&\int_{\partial L\cap\partial L_\varepsilon} \frac{\phi(\lambda)}{\sqrt{z-\lambda}}\, d\lambda - \int_{-\pi/2}^{\pi/2}\frac{\phi(z+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}\, d\theta}{\sqrt{\varepsilon e^{i\theta}}}\\ &- \int_{-\infty}^{0}\frac{\phi(t-i\varepsilon+z)}{\sqrt{-t+i\varepsilon}}\, dt + \int_{-\infty}^{0}\frac{\phi(t+i\varepsilon+z)}{\sqrt{-t-i\varepsilon}}\, dt\Bigg].\end{aligned}$$

The first integral is 0, since $\phi \in C_c^\infty(L)$. As $\varepsilon\downarrow 0$ the second integral converges to 0, while the square roots in the third and fourth integrands converge to $\sqrt{-t}$ and $-\sqrt{-t}$, respectively. Thus

$$\int_L \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda) = i\int_{-\infty}^{0}\frac{\phi(t+z)}{\sqrt{-t}}\, dt.$$

Let $\phi(x, y) = \psi(x)\alpha(y)$, where ψ and α are in $C_c^\infty(R)$. Then

$$(7)\qquad \int_L \frac{\bar{\partial}\phi(\lambda)}{\sqrt{z-\lambda}}\, dm(\lambda) = i\alpha(y)\int_0^\infty \frac{\psi(x-t)}{\sqrt{t}}\, dt.$$

Fix $A > \max\{|\operatorname{Re} z| : z \in L\}$. Notice that

$$(8)\qquad \int_0^\infty (x-t)^n\chi_{\{z:-A<z<A\}}(x-t)\frac{dt}{\sqrt{t}} = \int_0^{x+A}(x-t)^n\frac{dt}{\sqrt{t}} \quad \text{for } |x| \leqslant A.$$

This last expression is $\sqrt{x+A}$ times a polynomial in x of degree n, $p_n(x)$. (The leading coefficient of $p_n(x)$ is $\int_0^1 u^{-1/2}(1-u)^n\, du > 0$.) Choose ψ's to approximate $t^n\chi_{(-A,A)}(t)$ pointwise boundedly. Combining (6), (7), and (8), we see that $f \perp \alpha(y)\sqrt{x+A}\,p_n(x)$ (since the support of μ is contained in L). Thus by the Stone-Weierstrass theorem $\sqrt{x+A}\,f = 0$ in $L^2(\mu)$, so $f = 0$ in $L^2(\mu)$. □

It is not difficult to replace the branch chosen for Theorem 2 by a more complicated one; say a Jordan arc which is piecewise smooth and rectifiable in L. In fact the choice of branch may depend on λ if the perturbation is smooth. Also $\sqrt{z-\lambda}$ may be replaced by $\sqrt[\alpha]{z-\lambda}$, $\alpha > 0$, or, for example, by $(z-\lambda)\log(z-\lambda)$. Since the total variation of μ is finite, the μ measure of horizontal lines is zero, except for at most a countable set. Thus the $\sqrt{z-\lambda}$'s needed for Theorem 2 may be restricted so that $\sqrt{z-\lambda}$ is analytic μ-a.e.

We prove a similar theorem where the branches of $\sqrt{\ }$ vary, but where the base of $\sqrt{\ }$ is fixed. For each $\alpha \in [0, 2\pi)$ we define a function $f_\alpha(z) = r^{1/2}e^{i\theta/2}$, where $z = re^{i\theta}$ is chosen so that $\alpha \leqslant \theta < 2\pi + \alpha$. For this theorem no assumption concerning b.p.e.'s is needed.

THEOREM 3. $\mathrm{sp}\{P^2(\mu) + f_\alpha(z)P^2(\mu)\colon \alpha \in [0, 2\pi)\}^- = L^2(\mu)$.

PROOF. Let $0 \leqslant \alpha < \beta < 2\pi$. Then

$$\frac{f_\beta(z) - f_\alpha(z)}{2} = \begin{cases} 0 & \text{if } \arg z \in [\beta, 2\pi + \alpha) \bmod 2\pi, \\ -f_0(z) & \text{if } \arg z \in [\alpha, \beta) \bmod 2\pi. \end{cases} \tag{9}$$

Denote by $\mathcal{L}$ the closed linear span of $\{P^2(\mu) + f_\alpha(z)P^2(\mu)\colon \alpha \in [0, 2\pi)\}$. From (9) $f_0(z)\chi_{\{z:\, \arg z \in [\alpha,\beta) \bmod 2\pi\}}(z)p(z) \in \mathcal{L}$ for every polynomial p and $0 \leqslant \alpha < \beta < 2\pi$. Thus by approximation $f_0(z)h(\arg z)z^n \in \mathcal{L}$ for every nonnegative integer n and every continuous function h on $[0, 2\pi]$ with $h(0) = h(2\pi)$. If m is a nonnegative integer take $h(\arg z) = z^m\bar{z}^n/|z|^{m+n}$; if m is a negative integer let $h(\arg z) = \bar{z}^{n-m}/|z|^{n-m}$. In either case

$$f_0(z)h(\arg z)z^n = f_0(re^{i\theta})r^n e^{im\theta} \in \mathcal{L},$$

where $z = re^{i\theta}$. By the Stone-Weierstrass theorem $\mathrm{sp}\{r^n e^{im\theta}\colon n \in N, m \in Z\}$ is dense in $C(K)$. Thus if $k \in L^2(\mu)$ satisfies $k \perp \mathcal{L}$, then $\overline{f_0(z)}k(z)\,d\mu$ is the zero measure. Hence $k\,d\mu = c\delta_0$. But $1 \in \mathcal{L}$ so $k = 0$ in $L^2(\mu)$. $\square$

It is clear that only a dense subset of α's in $[0, 2\pi)$ is needed for Theorem 3. Again a similar argument holds for $z\log_\alpha z$, where $\log_\alpha(z) = \ln(z) + i\arg z$ and $\alpha \leqslant \arg z < \alpha + 2\pi$. Also a smooth one parameter family of nonintersecting smooth Jordan arcs emanating from a base point to ∞ can replace the radial lines.

Note. (a) The case $g(z) = \bar{z}$ of Theorem 1 was independently discovered and orally communicated to us by J. Thomson and R. Olin.

(b) When K is simply connected with empty interior, arguments involving b.p.e.'s and a result analogous to Theorem 2 lead to a new proof of a theorem of Lavrentieff on the uniform approximation of continuous functions on K by polynomials.

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