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A GENERALIZATION OF A THEOREM ON NATURALLY REDUCTIVE HOMOGENEOUS SPACES

OLDŘICH KOWALSKI AND LIEVEN VANHECKE

ABSTRACT. We prove that a homogeneous Riemannian manifold all of whose geodesics are orbits of one-parameter subgroups of isometries has volume-preserving local geodesic symmetries.

Let (M, g) be a Riemannian manifold. (M, g) is said to be *naturally reductive* [**6**] if there is a connected Lie group G of isometries acting transitively and effectively on M such that $M = G/H$ (H being the isotropy subgroup of G at some point of M) is a reductive homogeneous space with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G and moreover

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle [X, Z]_{\mathfrak{m}}, Y\rangle = 0 \quad \text{for } X, Y, Z \in \mathfrak{m}.$$

Here $\langle\ ,\ \rangle$ denotes the inner product on $\mathfrak{m}$ induced by the metric g. D'Atri and Nickerson [**1, 2**] have proved

THEOREM A. *Let (M, g) be a naturally reductive homogeneous space. Then*
(a) *all the local geodesic symmetries are volume-preserving;*
(b) *the characteristic polynomial $P(\lambda) = \det(g_{ij} - \lambda\delta_{ij})$ of the metric tensor g with respect to any system of normal coordinates centered at a point of the manifold is preserved under the local geodesic symmetry centered at the same point.*

In [**4, 5**] A. Kaplan introduced the notion of a *generalized Heisenberg group* and proved that these spaces are naturally reductive if and only if the center of the corresponding algebra has dimension 1 or 3. Moreover the following is true.

PROPOSITION B. *Let M be a generalized Heisenberg group. Then*
(a) *all the local geodesic symmetries are volume-preserving;*
(b) *if $\dim M = 6$ (two-dimensional center), then $P(\lambda)$ is preserved under the corresponding local geodesic symmetry at each point.*

(a) is proved in [**5**] and (b) in [**9**].

Next, let $(M, g) = G/H$ where G is the connected component $I_0(M)$ of the full isometry group of M. Then, (M, g) is said to be a *commutative space* if the algebra $\mathcal{D}(G/H)$ of invariant differential operators on G/H is commutative. The 6-dimensional Heisenberg group provides an example and hence, (b) of Proposition B is a corollary of the following theorem.

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THEOREM C [**7**]. *Let (M, g) be a commutative space. Then, the volume and also the characteristic polynomials $P(\lambda)$ are preserved under local geodesic symmetries.*

The main purpose of this paper is to prove two theorems which provide new proofs of Theorem A and (b) of Proposition B, and which at the same time generalize both.

THEOREM 1. *Let $(M, g) = G/H$ be a homogeneous Riemannian manifold such that all its geodesics are orbits of one-parameter subgroups of G. Then (M, g) is a space with volume-preserving geodesic symmetries.*

PROOF. For any fixed $m \in M$ and a normal neighborhood U_m, define the normal volume function $\theta_m : U_m \to \mathbf{R}$ by

$$\theta_m(p) = \omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)\bigg|_p, \qquad p \in U_m, \tag{1}$$

where ω is a (local) Riemannian volume form defined near m and $(x^1, \ldots, x^n)$ is a positively oriented system of normal coordinates centered at m. It is easy to see that $\theta_m(p)$ does not depend on the choice of ω and the choice of the normal coordinates.

Further we have, for each isometry $\varphi \in I(M)$,

$$\theta_{\varphi(m)}(\varphi(p)) = \theta_m(p) \tag{2}$$

and it is proved in [**3**, pp. 156–157] that

$$\theta_m(p) = \theta_p(m), \tag{3}$$

for any p sufficiently close to m.

Finally, let $-p$ denote the antipodal point of p with respect to the center m. Then, M is a space with volume-preserving geodesic symmetries if and only if for all m and all p near m we have

$$\theta_m(-p) = \theta_m(p). \tag{4}$$

To prove the required result, consider the geodesic $\gamma(t)$ such that $\gamma(0) = m$, $\gamma(\tau) = p$, $\gamma(-\tau) = -p$ for some $\tau > 0$. According to our assumption, there is an $A \in \mathfrak{g}$ such that $\gamma(t) = (\exp tA)(m)$ for all $t \in \mathbf{R}$. Put $\varphi = \exp(-\tau A)$. Then we have $\varphi(m) = -p$, $\varphi(p) = m$. Hence (2) and (3) imply (4).

THEOREM 2. *Under the hypothesis of Theorem 1, the characteristic polynomial $P(\lambda)$ of the metric tensor with respect to any normal coordinate system is preserved under the corresponding local geodesic symmetry.*

PROOF. We use the same argument as before; instead of (2) we have the invariance of $P(\lambda)$ with respect to all isometries and instead of (3) we use the formulas

$$\mu_{km}(p) = \mu_{kp}(m), \qquad k = 1, \ldots, n,$$

from [**10**], where the μ_{km} denote the coefficients of $P(\lambda)$ with reference point m.

Theorem A follows now at once since all geodesics of a naturally reductive homogeneous space are orbits of one-parameter subgroups. The same property holds for the six-dimensional generalized Heisenberg group (see [**5, 9**]) and hence (b) of

Proposition B also follows. In fact, Theorem 2 applies to the broader subclass of generalized Heisenberg groups determined explicitly by C. Riehm (see [**5**, p. 42]).

REMARK. It can be seen from [**5**, **8**] that there are generalized Heisenberg groups (with four-dimensional center) whose geodesics are not all orbits of one-parameter subgroups but which are still commutative spaces. On the other hand, there are also generalized Heisenberg groups (for example with five-dimensional center) whose geodesics are all orbits of one-parameter subgroups but it is not known to us whether these spaces are commutative or not. Yet, we believe that the property "geodesics are orbits of one-parameter subgroups" might well imply the commutativity for the class of all Riemannian homogeneous manifolds.

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FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 18600 PRAHA, CZECHOSLOVAKIA

DEPARTMENT OF MATHEMATICS, KATHOLIEKE UNIVERSITEIT LEUVEN, CELESTIJNENLAAN 200B, B-3030 LEUVEN, BELGIUM