ON PARTIALLY CONSERVATIVE SENTENCES
AND INTERPRETABILITY

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Abstract. A sentence \( \varphi \) is \( \Gamma \)-conservative over \( T \) if \( T + \varphi \vdash \psi \) implies \( T \vdash \psi \) for every \( \psi \in \Gamma \). In §1 this concept for \( \Gamma = \Sigma^0_{n+1} \) and \( \Pi^0_{n+1} \) is investigated. In §2 results from §1 are applied to interpretability in theories containing arithmetic.

0. Introduction. Let \( \Gamma \) be a set of sentences. A sentence \( \varphi \) is \( \Gamma \)-conservative over a theory \( T \), \( \varphi \in \text{Cons}(\Gamma, T) \), if \( T + \varphi \vdash \psi \) implies \( T \vdash \psi \) for every \( \psi \in \Gamma \). This concept was introduced by Guaspari [2] and the basic existence theorems were established by him and Solovay (cf. [2]). The first example of a result of this type is, however, due to Kreisel [7] who observed that if \( \text{Con}_P \) is a “natural” formalization of “\( P \) (Peano arithmetic) is consistent”, then \( \neg \text{Con}_P \in \text{Cons}(\Pi^0_1, P) \). Related results have also been obtained by Hájek [4], Jensen and Ehrenfeucht [6], and Kreisel and Lévy [8]. Subsequently the results of Guaspari and Solovay were somewhat improved and their proofs were simplified by Hájek [4], Smoryński [14, 15], and the author. The general method of proof applied in the papers mentioned as well as in the present paper is, however, the same, namely a combination of partial truth definitions and self-reference.

In §1 we present two groups of results. Those belonging to the first group (Theorems 1–5 and Corollary 1) concern the existence of partially conservative sentences and partially conservative extensions of theories satisfying various additional conditions. In the results of the second group (Theorems 6 and 7 and Corollaries 2 and 3) it is shown that a number of sets occurring naturally in the study of partial conservativity are complete at certain levels of the arithmetical hierarchy. In §2 results from §1 are applied to (relative) interpretability using the fact that if \( S \) and \( T \) are reflexive r.e. extensions of \( P \), then \( S \) is interpretable in \( T \) iff every \( \Pi^0_1 \) sentence provable in \( S \) is provable in \( T \). In particular we answer two questions raised by Orey [12] and Švejdar [17]. For further applications of partially conservative sentences to interpretability see [10, 11].

1. Partially conservative sentences. In the following \( \Gamma \) is either \( \Sigma^0_{n+1} \) or \( \Pi^0_{n+1} \). \( \check{\Gamma} \) is the dual of \( \Gamma \). \( A \) and \( B \) are consistent primitive recursive extensions of \( P \). (By Craig’s theorem (cf. [1]), every r.e. set of (\( \Gamma \)) sentences is deductively equivalent to a primitive recursive set of (\( \Gamma \)) sentences.) \( \text{Th}(T) = \langle \varphi: T \vdash \varphi \rangle \). We write \( T \vdash X \) or

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\[ X \vdash T \text{ to mean that } X \subseteq \text{Th}(T). \]  
\[ S \text{ is an } X\text{-subtheory of } T, \quad S \vdash_X T, \quad \text{if } \text{Th}(S) \cap X \subseteq \text{Th}(T). \]
\[ \varphi' \text{ is } \varphi \text{ if } i = 0 \text{ and } \neg \varphi \text{ if } i = 1. \]

The diagonal lemma (self-reference) will be used repeatedly without further comment. For notation and terminology not explained here see [1].

As is well known, to each \( \Gamma \) there is a \( \Gamma \) formula \( \Gamma\text{-true}(\varphi) \) s.t. for every \( \varphi \in \Gamma \),
\[ P \vdash \varphi \leftrightarrow \Gamma\text{-true}(\varphi). \]

Let \( \alpha(x) \) be a PR binumeration of \( A \) and let \( (\Gamma)(x, y) \) be the formula
\[ \forall uv \leq y\left(u \text{ is } \Gamma \land \text{Prf}_{\alpha(z)} \lor z \quad \rightarrow \quad \Gamma\text{-true}(u)\right) \]
(cf. [2, 4, 6, 8, 14]). The following lemma is then easily verified.

**Lemma 1.** \( (\Gamma)(x, y) \) is a \( \Gamma \) formula s.t.
(i) \( P \vdash (\Gamma)(x, y) \land y' \leq y \rightarrow (\Gamma)(x, y'), \)
(ii) \( A + \varphi \vdash (\Gamma)(\varphi, m), \) all \( \varphi \) and \( m, \)
(iii) if \( \psi \) is \( \Gamma \) and \( A + \varphi \vdash \psi \), then there is a \( q \) s.t. \( P + (\Gamma)(\varphi, q) \vdash \psi. \)

The next two lemmas serve to unify a number of proofs in what follows.

**Lemma 2.** Suppose \( \chi(x, y) \) is \( \Gamma \). Then there is a \( \Gamma \) formula \( \xi(x) \) s.t.
(i) \( A + \chi(k, m) \vdash \chi(k, m), \)
(ii) if \( B \vdash A \), then \( B + \xi(k) \vdash B \cup \chi(k, q) \) \( q \in \omega \).

**Proof.** Case 1. \( \Gamma = \Pi^0_{n+1} \). Let \( \xi(x) \) be s.t.
\[ P \vdash \xi(k) \leftrightarrow \forall y\left( (\Sigma^0_{n+1})(\xi(k), y) \rightarrow \chi(k, y) \right). \]
Then (i) follows at once from Lemma 1(ii). To prove (ii), suppose \( \psi \) is \( \Sigma^0_{n+1}, B \vdash A, \) and \( B + \xi(k) \vdash \psi. \) By Lemma 1(iii), there is a \( q \) s.t.
\[ P + (\Sigma^0_{n+1})(\xi(k), q) \vdash \psi. \]
Hence, by Lemma 1(i),
\[ P + \forall y \leq \xi(k) \chi(k, y) + \neg \psi \vdash \xi(k). \]
But then, since \( B + \xi(k) \vdash \psi \), it follows that \( B + \forall y \leq \xi(k) \chi(k, y) \vdash \psi. \)

Case 2. \( \Gamma = \Sigma^0_{n+1} \). Let \( \xi(x) \) be s.t.
\[ P \vdash \xi(k) \leftrightarrow \exists y\left( \neg (\Pi^0_{n+1})(\xi(k), y) \land \forall z \leq y \chi(k, z) \right). \]
The proof that \( \xi(x) \) is as claimed is almost the same as in Case 1.

**Lemma 3.** Suppose \( \chi_0(x, y) \) is \( \Gamma \) and \( \chi_1(x, y) \) is \( \Gamma \). Then there is a \( \Gamma \) formula \( \xi(x) \) s.t. for \( i = 0, 1 \)
(i) \( A + \xi(k) \vdash \forall y \leq m \chi_i(k, y) \rightarrow \chi_{i-1}(k, m), \)
(ii) if \( \psi \) is \( \Gamma \) and \( A + \xi(k) \vdash \psi \), then \( A \cup \chi_{i-1}(k, q) \) \( q \in \omega \) \vdash \psi.

**Proof.** We need only prove this for \( \Gamma = \Sigma^0_{n+1} \). Let \( \xi(x) \) be s.t.
\[ P \vdash \xi(k) \leftrightarrow \exists y\left( (\neg (\Pi^0_{n+1})(\xi(k), y) \lor \neg \chi_0(k, y) ) \land \forall z < y\left((\Sigma^0_{n+1})(\neg \xi(k), z) \land \chi_1(k, z) \right)\right). \]
The proof of Lemma 3 is now very much the same as the proof of Lemma 2.
Let us say that \( \varphi \) is hereditarily \( \Gamma \)-conservative over \( A \), \( \varphi \in \text{HCons}(\Gamma, A) \), if \( \varphi \in \text{Cons}(\Gamma, B) \) for every \( B \vdash A \). The following result is due to Guaspari [2] (cf. also [4, 14, 15]).

**Theorem 1.** Let \( X \) be any r.e. set. There is then a \( \Gamma \) formula \( \xi(x) \) s.t.

(i) if \( k \in X \), then \( A \vdash \neg \xi(k) \),
(ii) if \( k \not\in X \), then \( \xi(k) \in \text{HCons}(\bar{\Gamma}, A) \).

**Proof.** Let \( p(x, y) \) be a PR binumeration of a relation \( R(k, m) \) s.t. \( X = \{k: \exists m R(k, m)\} \) and let \( \xi(x) \) be as in Lemma 2 with \( \chi(x, y) = \neg p(x, y) \). Then (i) follows from Lemma 2(i) and (ii) follows from Lemma 2(ii).

A set \( X \) of sentences is mono-consistent with \( T \) if \( T + \varphi \) is consistent for every \( \varphi \in X \) (cf. [9]).

**Corollary 1.** If \( X \) is r.e. and mono-consistent with \( A \), then \( \Gamma \cap \text{HCons}(\bar{\Gamma}, A) - X = \emptyset \).

**Proof.** Let \( \xi(x) \) be as in Theorem 1 and let \( \varphi \) be s.t. \( P \vdash \varphi \iff \xi(\overline{\varphi}) \). If \( \varphi \in X \), then \( A \vdash \neg \xi(\overline{\varphi}) \), whence \( A \vdash \neg \varphi \). But this is impossible and so \( \varphi \notin X \). But then, by Theorem 1(ii), \( \varphi \) is as desired.

Let \( NX = \{\varphi: \neg \varphi \in X\} \) and \( \text{DCons}(\Gamma, A) = \text{Cons}(\Gamma, A) \cap \text{NCons}(\bar{\Gamma}, A) \). Solovay proved that \( \Gamma \cap \text{DCons}(\bar{\Gamma}, A) = \emptyset \) (cf. [2]). This can be improved as follows (cf. [14, 15]).

**Theorem 2.** Suppose \( X \) is r.e. and mono-consistent with \( A \). Then \( \Gamma \cap \text{DCons}(\bar{\Gamma}, A) = (X \cup NX) = \emptyset \).

**Proof.** Let \( p_i(x, y) \) be PR binumerations of relations \( R_i(k, m) \) s.t. \( X = \{k: \exists m R_0(k, m)\} \) and \( NX = \{k: \exists m R_i(k, m)\} \). Let \( \xi(x) \) be as in Lemma 3 with \( \chi_i(x, y) = \neg p_i(x, y) \) and let \( \theta \) be s.t. \( P \vdash \theta \iff \xi(\overline{\theta}) \). Suppose \( \theta \in X \cup NX \). Let \( n \) be the least number s.t. \( R_0(\theta, n) \) or \( R_i(\theta, n) \). Suppose \( R_i(\theta, n) \). Then not \( R_{i-1}(\theta, m) \) for \( m < n \). (We may assume that \( R_0(k, m) \) implies not \( R_i(k, m) \).) Hence, by Lemma 3(ii), \( A \vdash \xi(\overline{\theta}) \), whence \( A \vdash \neg \theta_i \), which is impossible since \( \theta_i \in X \). Thus \( \theta \notin X \cup NX \). But then, by Lemma 3(ii), \( \theta \in \text{DCons}(\bar{\Gamma}, A) \).

The result of Solovay mentioned above together with Corollary 1 with \( X = \text{Th}(A) \) led to the question if \( Y = \cap \{\text{DCons}(\Gamma, B): B \vdash A\} = \emptyset \) (cf. [2, 14]). It is easily seen, however, that this is not true in general. Let \( \varphi \) be a \( \Gamma \) sentence and \( \psi \) a \( \bar{\Gamma} \) sentence s.t. \( P \vdash \varphi \lor \psi \), \( i = 0, 1 \), and let \( A = P + \varphi \land \psi \). Suppose \( \theta \in Y \). Then, since \( P + \theta \vdash \varphi + \theta \vdash \varphi \), we get \( P + \theta \vdash \varphi \lor \psi \) and so \( P + \neg \theta \vdash \varphi \). Similarly \( P + \theta \vdash \psi \). But then \( P \vdash \varphi \lor \psi \), contrary to hypothesis.

Our next result will be applied in §2 to answer a question of Orey [12].

**Theorem 3.** There are sentences \( \varphi_i \) s.t. \( \varphi_i, \neg(\varphi_0 \land \varphi_1) \in \text{Cons}(\Gamma, A) - \text{NCons}(\bar{\Gamma}_i, A) \), \( i = 0, 1 \).

We write \( A \vdash^p \chi B \) ("\( p \)" for "proper") to mean that \( A \vdash^p \chi B \not\equiv \chi A \).

**Lemma 4.** Suppose \( A \vdash B \equiv_{\Pi_1^0} A \). Then there is a sentence \( \chi \) s.t. \( A \vdash \Pi_1^0 A + \chi \vdash B \), \( i = 0, 1 \).
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Proof. Let $\theta$ be a $\Pi^0_1$ sentence s.t. $B \vdash \theta$ and $A \not\vdash \theta$ and let $\theta_i$ be s.t.

$$P \vdash \theta_i \iff \forall z (\Prf_n(\theta \lor \theta_i, z) \to \exists u \leq i z \Prf_n(\theta \lor \theta_{i-1}, u)),$$

where $\leq_0$ is $\leq$ and $\leq_1$ is $< (cf. [6])$. Then, by a standard argument,

1. $P \vdash \theta_0 \lor \theta_i$.
2. $A \not\vdash \theta \lor \theta_i$.

Suppose $\Gamma \cup \Gamma \subseteq \Gamma^*$. By Lemma 3, there is a sentence $\chi$ s.t.

3. $A \vdash \chi' \vdash (\theta \lor \theta_i) \to (\theta \lor \theta_{i-1})$.
4. if $\psi$ is $\Gamma^*$ and $A \vdash \chi' \vdash \psi'$, then $A \vdash (\theta \lor \theta_{i-1}) \vdash \psi'$.

By (1), (2), (3), $A \vdash \Pi^0_1 A + \chi'$. Finally, by (4), $A + \chi' \vdash \Gamma B$.

Proof of Theorem 3. Let $\eta$ and $\theta$ be s.t. $\eta' \not\in \text{Cons}(\Pi^0_1, A)$ and $\theta' \in \text{Cons}(\Gamma, A)$, $i = 0, 1$ (cf. [17] and Theorem 2). Let $\psi = \eta \land \theta$. Then

1. $\psi \in \text{NCons}(\Gamma, A) - \text{Cons}(\Pi^0_1, A)$,
2. $\psi \lor \neg \eta \in \text{Cons}(\Gamma, A)$.

Moreover $A \vdash \Pi^0_1 A + \neg \eta$. So, by Lemma 4, there is a sentence $\chi$ s.t.

3. $A \vdash \Pi^0_1 A + \chi' \vdash \Gamma A + \neg \eta$.

Finally let $\psi_i = \psi \land \chi'$. Then, by (2) and (3), $\psi_i \in \text{Cons}(\Gamma, A) - \text{NCons}(\Pi^0_1, A)$.

Moreover $\vdash (\psi_0 \land \psi_1) \iff \psi$. So, by (1), $\psi_0$ and $\psi_1$ are as desired.

The next two results are refinements of the following simple and certainly well-known observation. Suppose $X$ is r.e., bounded, i.e. $X \subseteq \Gamma$ for some $\Gamma$, and $A \cup X$ is consistent. Then there is a sentence $\theta$ s.t. $A \cup X \vdash A + \theta$ and $A + \theta$ is consistent. This can be improved as follows. Let us say that $S$ is a $\Gamma$-conservative extension of $T$ if $T \vdash S \vdash \Gamma B$.

Theorem 4. Let $X$ be an r.e. set of $\Gamma$ sentences. Then there is a $\Gamma$ sentence $\theta$ s.t.

$A + \theta$ is a $\Gamma$-conservative extension of $A \cup X$.

Proof. By Craig's theorem, we may assume that $X$ is primitive recursive. Let $\xi(x)$ be a PR binumeration of $X$. Then

1. $P \cup X \vdash m \vdash \forall z \leq m(\xi(z) \to \Gamma \text{-true}(z))$.

By Lemma 2, there is a $\Gamma$ sentence $\theta$ s.t.

2. $A + \theta \vdash \xi(\bar{q}) \to \Gamma \text{-true}(\bar{q})$,
3. $A + \theta \vdash \bar{r} A \cup \langle \xi(\bar{q}) \to \Gamma \text{-true}(\bar{q}): q \in \omega \rangle$.

From (2) it follows that $A + \theta \vdash X$. Suppose $\psi$ is $\Gamma$ and $A + \theta \vdash \psi$. Then, by (1) and (3), $A \cup X \vdash \psi$. Thus $A + \theta \vdash \bar{r} A \cup X$.

A closely related result is the following

Theorem 5. Let $X$ be an r.e. set of $\Gamma$ sentences and let $Y$ be any r.e. set s.t. $A \cup X \not\vdash \psi$ for every $\psi \in Y$. Then there is a $\Gamma$ sentence $\theta$ s.t. $A \cup X \vdash A + \theta \not\vdash \psi$ for every $\psi \in Y$.

Proof. We may assume that $X$ and $Y$ are primitive recursive. Let $\xi(x)$ and $\eta(x)$ be PR binumerations of $X$ and $Y$, respectively.

Case 1. $\Gamma = \Pi^0_{n+1}$. Let $\theta$ be s.t.

$$P \vdash \theta \iff \forall y \langle \xi(y) \land \forall z u \leq y(\eta(z) \to \neg \Prf_{\alpha(x) \lor x = \delta(z, u)} \to \Pi^0_{n+1} \text{-true}(y)) \rangle.$$
Suppose $\psi \in Y$ and $A + \theta \vdash \psi$. Then $P \cup X \vdash \theta$, whence $A \cup X \vdash \psi$, contrary to hypothesis. Thus $A + \theta \not\vdash \psi$ for $\psi \in Y$. But then it follows that $A + \theta \vdash X$.

**Case 2.** $\Gamma = \Sigma_{n+1}$. Let $\theta$ be s.t.

$$P \vdash \theta \iff \exists y (\exists z u \leq y (\eta (z) \land \text{Prf}_{n+1} (z, u)) \land \forall z \leq y (\xi (z) \rightarrow \Sigma_{n+1} \text{true}(z))).$$

The verification that $\theta$ is as claimed is now straightforward.

Guaspari [2] observed that Cons($\Gamma$, $A$) is $\Pi^0_2$ and that Theorem 1 implies that $\Gamma \cap \text{Cons}(\Gamma, A)$ is not r.e. and suggested the problem of classifying these sets. A complete solution (and more) follows from our next theorem. Partial results have been obtained by Hájek [4], Quinsey [13] and Solovay [16] using more complicated methods (cf. also [14]). Let

$$\text{Cons}(\Gamma, Y, A) = \{ \phi : \text{for every } \psi \in \Gamma, \text{ if } A + \phi \vdash \psi, \text{ then } \psi \in Y \}.$$

**Theorem 6.** Suppose $\Gamma = \Pi^0_1$ and $Y$ is r.e. and mono-consistent with $P$. Then to any $\Sigma^0_1$ set $X$, there is a $\Gamma$ formula $\xi (x)$ s.t.

(i) if $k \in X$, then $\xi (k) \in \text{HCons}(\Gamma, A),$  
(ii) if $k \notin X$, then $\forall r < p, \xi (k, r) \notin \text{Cons}(\Pi^0_1, Y, A) \cup \text{Cons}(\Sigma^0_1, Y, A).$

To prove this we need the following

**Lemma 5.** Suppose $X$ and $Y$ are r.e. and $Y$ is mono-consistent with $P$. Then there is a $\Sigma^0_1$ formula $\xi_0 (x)$ and a $\Pi^0_1$ formula $\xi_1 (x)$ s.t.

(i) $P \vdash \xi_0 (k) \rightarrow \xi_1 (k)$,  
(ii) if $k \in X$, then $P \vdash \xi_0 (k)$,  
(iii) if $k \notin X$, $r < p$, then $\forall r < p, \xi_0 (k, r) \notin Y.$

**Proof.** Let $\rho (x, y)$ and $\sigma (x, y)$ be PR binumerations of relations $R(k, m)$ and $S(k, m)$ s.t. $X = \langle k : \exists m R(k, m) \rangle$ and $Y = \langle k : \exists m S(k, m) \rangle$ and let $\xi_1 (x)$ be s.t.

$$P \vdash \xi_1 (k) \leftrightarrow \forall u (\chi (u) \rightarrow \exists z \leq u \rho (k, z)),$$

where $\chi (u)$ is the formula

$$\exists v \leq u (\sigma (v, u) \land \text{"v is of the form } \forall r < p \xi_0 (k, r) \land \forall w \leq u \forall q \leq p \rightarrow \rho (k_q, w)).$$

Finally let $\xi_0 (x)$ be s.t.

$$P \vdash \xi_0 (k) \leftrightarrow \exists z (\rho (k, z) \land \forall u \leq z \rightarrow \chi (u)).$$

Then (i) is immediate. Next we show that for every $m$, $\chi (m)$ is false. Suppose $\chi (m)$ is true. Then

$$P \vdash \xi_1 (k) \rightarrow \exists z \leq m \rho (k, z).$$

Moreover there are $k_r$, $r < p$, s.t. $\forall r < p, \xi_1 (k_r) \in Y$ and $P \vdash \neg \rho (k_r, s)$ for $r < p$ and $s \leq m$. But then $P \vdash \neg \forall r < p, \xi_1 (k_r)$, a contradiction. Thus $\chi (m)$ is false. But then (ii) and (iii) follow at once, the latter, since we may assume that $S(m, n)$ only if $m \leq n$. 

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Proof of Theorem 6. We may assume that if \( \varphi \in Y \) and \( P \vdash \varphi \rightarrow \psi \), then \( \psi \in Y \). Let \( R(k, m) \) be an r.e. relation s.t. \( X = \{k: \forall m\ R(k, m)\} \). By Lemma 5, there is a \( \Sigma^0_1 \) formula \( \rho_0(x, y) \) and a \( \Pi^0_1 \) formula \( \rho_1(x, y) \) s.t.

1. \( P \vdash \rho_0(k, m) \rightarrow \rho_1(k, m) \),
2. if \( R(k, m) \), then \( P \vdash \rho_0(k, m) \),
3. if not \( R(k, m) \), \( m \leq p \), then \( \forall m \leq p \rho_1(k, m) \notin Y \).

By Lemma 2, there is a \( \Gamma \) formula \( \xi(x) \) s.t.

4. \( \forall x \xi(x) \vdash \rho_0(k, m) \),
5. if \( B \vdash \varphi \), then \( B + \xi(k) \vdash \xi\overline{B} \cup \{\rho_0(k, \bar{q}): q \in \omega\} \).

Now (i) follows from (2) and (5) and (ii) follows from (1), (3) and (4).

Corollary 2. The sets \( \Gamma \cap \text{Cons}(T, A) \), where \( \Gamma = \Pi^0_1 \cap \text{Cons}(\Sigma^0_1, A) \), and \( \Sigma^0_2 \cap \text{Cons}(\Sigma^0_1, A) \) are complete \( \Pi^0_2 \) sets.

If \( A \) is \( \Sigma^0_1 \)-sound, then \( \Pi^0_1 \cap \text{Cons}(\Sigma^0_1, A) \) is \( \Pi^0_1 \). However, Quinsey [13] has shown that if \( A \) is not \( \Sigma^0_1 \)-sound, then this set is a complete \( \Pi^0_2 \) set.

There are results similar to Theorem 6 and Corollary 2 for the sets \( \Gamma \cap \text{DCons}(T, A) \) but at this point they are quite easy and are therefore omitted.

The following corollary is applied in [10].

Corollary 3. If \( \Gamma = \Pi^0_1 \), then \( Z = \{\varphi: \exists y \in \Gamma \cap \text{Cons}(T, A) \ (A + \varphi \vdash \varphi)\} \) is a complete \( \Sigma^0_3 \) set.

Proof. Clearly \( Z \) is \( \Sigma^0_3 \). Let \( X \) be any \( \Sigma^0_3 \) set and let \( R(k, m) \) be a \( \Pi^0_2 \) relation s.t. \( X = \{k: \exists m\ R(k, m)\} \). By Theorem 6, there is a \( \Gamma \) formula \( \rho(x, y) \) s.t.

1. if \( R(k, m) \), then \( \rho(k, m) \in \text{Cons}(T, A) \),
2. if not \( R(k, m) \), \( m \leq p \), then \( \forall m \leq p \rho(k, m) \notin \text{Cons}(T, A) \).

Let \( Y_k = \{-\rho(k, m): m \in \omega\} \). Then, by (the proof of) Theorem 4, there is a formula \( \eta(x) \) s.t. \( A \cup Y_k \vdash A + \eta(k) \rightarrow \overline{A} \cup Y_k \). It follows that \( X = \{k: \eta(k) \in Y\} \) and so the proof is complete.

Suppose \( X \) is r.e. and let \( Y = \{\varphi: A + \varphi \vdash X\} \). Clearly \( Y \) is r.e. unless \( X \) is infinite over \( A \) in the sense that \( A \cup X \upharpoonright k \not\vdash X \) for every \( k \).

Theorem 7. Suppose \( X \) is r.e., bounded, and infinite over \( A \). Then \( Y = \{\varphi: A + \varphi \vdash X\} \) is a complete \( \Pi^0_2 \) set.

This was proved independently by Christian Bennet. In fact, the proof below is due to him and is presented here with his permission.

Proof. As usual we may assume that \( X \) is primitive recursive. Let \( \xi(x) \) be a PR binumeration of \( X \). Let \( n \) be s.t. \( X \subseteq \Pi^0_{n+1} \). Next let \( Z \) be any \( \Pi^0_2 \) set and let \( R(k, m) \) be an r.e. relation s.t. \( Z = \{k: \forall m\ R(k, m)\} \). By Theorem 1, there is a formula \( \rho(x, y) \) s.t.

1. if \( R(k, m) \), then \( A \vdash \rho(k, m) \),
2. if not \( R(k, m) \), then \( -\rho(k, m) \in \text{Cons}(\Sigma^0_{n+2}, A) \).

Let \( \eta(x) \) be the formula

\[ \forall z (\xi(z) \wedge \forall u \leq z \rho(x, u) \rightarrow \Pi^0_{n+1}\text{-true}(z)). \]

It suffices to show that \( Z = \{k: \eta(k) \in Y\} \). If \( k \in Z \), then, by (1), \( A + \eta(k) \vdash \xi(\varphi) \rightarrow \Pi^0_{n+1}\text{-true}(\varphi) \) for every \( \varphi \), whence \( \eta(k) \in Y \). Next suppose \( k \notin Z \). Let \( m \) be s.t. \( -\rho(k, m) \notin \text{Cons}(\Sigma^0_{n+2}, A) \). By Theorem 6, there is a formula \( \rho(x, y) \) s.t.

1. if \( R(k, m) \), then \( A \vdash \rho(k, m) \),
2. if not \( R(k, m) \), then \( -\rho(k, m) \in \text{Cons}(\Sigma^0_{n+2}, A) \).

Let \( \eta(x) \) be the formula

\[ \forall z (\xi(z) \wedge \forall u \leq z \rho(x, u) \rightarrow \Pi^0_{n+1}\text{-true}(z)). \]
not $R(k, m)$. Then $A \cup X \vdash m \vdash \rho(\overline{k}, \overline{m}) \vdash \eta(\overline{k})$. Let $\varphi \in X$. If $A + \eta(\overline{k}) \vdash \varphi$, then $A + \vdash \rho(\overline{k}, \overline{m}) \vdash X \vdash m \rightarrow \varphi$, whence, by (2), $A \cup X \vdash m \vdash \varphi$. But, $X$ being infinite over $A$, this fails for some $\varphi \in X$. It follows that $\eta(\overline{k}) \notin Y$.

2. Applications to interpretability. We write $S \models T$ to signify that $S$ is interpretable in $T$. Most applications of partially conservative sentences to interpretability are based on the following (cf. [2, 3, 9])

**Lemma 6.** If $S$, $T$ are r.e. reflexive extensions of $P$, then $S \models T$ iff $S \vdash \Pi_P T$.

For brevity we assume from now on that $A$ and $B$ are essentially reflexive.

From Corollary 1 and Lemma 6 we get at once the following result essentially due to Hájek [3] (cf. also [5, 9, 14, 15, 16]).

**Theorem 8.** If $X$ is r.e. and mono-consistent with $A$, then there is a $\Sigma_0^1$ sentence $\varphi$ s.t. $A + \varphi \models A$ and $\varphi \notin X$.

Similarly, Theorem 2 yields the following (cf. [9])

**Theorem 9.** If $X$ is r.e. and mono-consistent with $A$, then there is a sentence $\theta$ s.t. $A + \theta^i \models A$ and $\theta^i \notin X$, $i = 0, 1$.

Theorem 8 was originally proved to give an example of a sentence $\varphi$ s.t. $ZF + \varphi \models ZF$ and $GB + \varphi \not\models GB$ and this follows at once, since $(\varphi$: $GB + \varphi \not\models GB$) is r.e., $GB$ being finite and mono-consistent with $ZF$. Similarly, Theorem 9 yields a sentence $\theta$ s.t. $ZF + \theta^i \models ZF$ and $GB + \theta^i \not\models GB$, $i = 0, 1$.

Our next result, which is an immediate consequence of Theorem 3, answers a question raised by Orey [12].

**Theorem 10.** There are sentences $\varphi_i$ s.t. $A + \varphi_i \models A$, $A + \varphi_0 \land \varphi_1 \not\models A$, $A + \neg \varphi_i \not\models A$, and $A + \neg \varphi_0 \lor \neg \varphi_1 \models A$, $i = 0, 1$.

Let $A = B$ mean that $A \models B$. Then from Theorem 4 we get ((i) is proved in [9])

**Theorem 11.** (i) If $A \models B$, then there is a sentence $\theta$ s.t. $A + \theta \models B$.
(ii) If $X$ is an r.e. set of $\Sigma_1^0$ sentences, then there is a $\Sigma_1^0$ sentence $\sigma$ s.t. $A + \sigma \equiv A \cup X$.

The following corollary answers a question suggested by Švejdar [17].

**Corollary 4.** There is a $\Sigma_1^0$ sentence $\sigma$ s.t. $A + \sigma \not\models A + \psi$ for every $\Pi_1^0$ sentence $\psi$.

**Proof.** There is an r.e. set $X$ of $\Sigma_1^0$ sentences s.t. $A \cup X \not\models A \cup X \vdash m$ for every $m$ (cf. [9]). By Theorem 11(ii), there is a $\Sigma_1^0$ sentence $\sigma$ s.t. $A + \sigma \equiv A \cup X$. Let $\psi$ be any $\Pi_1^0$ sentence and suppose $A + \sigma \equiv A + \psi$. Then, by Lemma 6, there is an $m$ s.t. $A \cup X \vdash m \vdash \psi$. But then $A \cup X \models A + \sigma \equiv A + \psi \not\models A \cup X \vdash m$, whence $A \cup X \not\models A \cup X \vdash m$, a contradiction.

The following result which was first proved by Solovay [16] (cf. also [2, 9, 14]) answers a question of Hájek [3]. It is an immediate consequence of Lemma 6 and Theorem 6 with $Y = \text{Th}(B)$.
Theorem 12. If $A \leq B$, then $\Sigma_1^0 \cap \{ \varphi : A + \varphi \leq B \}$ is a complete $\Pi_2^0$ set.

If there is a $\Pi_1^0$ sentence $\theta$ s.t. $B \vdash \theta$ and $B \leq A + \theta$, then, by Lemma 6, $\{ \varphi : B \leq A + \varphi \}$ is r.e. In contrast to this we have (cf. [9])

Theorem 13. If there is no $\Pi_1^0$ sentence $\theta$ s.t. $B \vdash \theta$ and $B \leq A + \theta$, then $\{ \varphi : B \leq A + \varphi \}$ is a complete $\Pi_2^0$ set.

Proof. By assumption, the set of $\Pi_1^0$ sentences provable in $B$ is infinite over $A$. Moreover $B \leq A + \varphi$ iff $A + \varphi \vdash \chi$. Now apply Theorem 7.

Suppose e.g. $\sigma$ is as in Corollary 4. Then, by Theorem 13, $\{ \varphi : A + \sigma \leq A + \varphi \}$ is a complete $\Pi_2^0$ set.

References

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