ON SPACES OF MAPS
BETWEEN COMPLEX PROJECTIVE SPACES

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ABSTRACT. When 1 < m < n, the space \( M(P^m, P^n) \) of maps of complex projective \( m \)-space \( P^m \) into complex projective \( n \)-space \( P^n \) has a countably infinite number of components enumerated by degrees of maps in \( H^2(P^m; \mathbb{Z}) \). By calculating their \((2n - 2m + 1)\)-dimensional integral homology group we show that two components of \( M(P^m, P^n) \) are homotopy equivalent if and only if their associated degrees have the same absolute value.

1. Introduction and statement of result. Let \( P^n \) denote the complex projective \( n \)-space. For 1 < m < n, consider the space \( M(P^m, P^n) \) of (continuous) maps of \( P^m \) into \( P^n \) equipped with the compact-open topology. \( M(P^m, P^n) \) has a countably infinite number of (path-)components, for the homotopy classes of maps of \( P^m \) into \( P^n \) are classified by their degrees in \( H^2(P^m; \mathbb{Z}) \equiv \mathbb{Z} \). For each integer \( k \in \mathbb{Z} \), let \( M_k(P^m, P^n) \) denote the component of \( M(P^m, P^n) \) consisting of maps of degree \( k \).

The object of this paper is to prove the following

THEOREM. The \((2n - 2m + 1)\)-dimensional integral homology group of \( M_k(P^m, P^n) \), 1 < m < n, is cyclic of infinite order for \( k = 0 \) and cyclic of order \( \binom{n+1}{m} |k|^m \) for \( k \neq 0 \):

\[
H_{2n-2m+1}(M_k(P^m, P^n)) \cong \mathbb{Z}/(n+1)|k|^m \mathbb{Z}.
\]

Some special cases of this result already occur in the literature. See [4, 2], for the case \( n = m = 1 \) and [5] for the case \( n = m \).

An easily derived consequence of the Theorem is the following

COROLLARY. Two components \( M_k(P^m, P^n) \) and \( M_l(P^m, P^n) \) of \( M(P^m, P^n) \) are homotopy equivalent if and only if \( |k| = |l| \).

This corresponds to result obtained by V. L. Hansen [1, 2, 3] for the case of spaces of maps of \( n \)-manifolds into the \( n \)-sphere.

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2. Restriction fibrations. For any two spaces \( X \) and \( Y \), we shall write \( M(X, Y) \) for the space of maps of \( X \) into \( Y \) equipped with the compact-open topology, and for
any map \( f: X \to X', \tilde{f}: M(X', Y) \to M(X, Y) \) denotes the map defined by composition with \( f; \) i.e. \( \tilde{f}(\alpha) = \alpha \circ f \) for \( \alpha \in M(X', Y) \).

Let \( E^{2m} \subset \mathbb{C}^m \) be the closed \( 2m \)-disc and \( \iota: S^{2m-1} = \partial E^{2m} \hookrightarrow E^{2m} \) the inclusion map. By restriction of maps to \( E^{2m} \), we obtain a fibration

\[
\tilde{\iota}: M(E^{2m}, P^n) \to M(E^{2m}, P^n)
\]

with \( \Omega^{2m}P^n \) as fibre.

Similarly, the inclusion \( \iota_{m-1}: P^{m-1} \hookrightarrow P^m \subset P^n, 1 < m \leq n \), induces a fibration

\[
\iota^k_{m-1}: M_k(P^m, P^n) \to M_k(P^{m-1}, P^n).
\]

This restriction fibration is the main object of study in this paper.

Since \( P^m = P^{m-1} \cup \partial E^{2m} \) is the mapping cone of the canonical projection \( p: E^{2m} = S^{2m-1} \to P^{m-1} \), the fibration \( \iota^k_{m-1} \) is the pull-back of \( \iota \) along the map \( \tilde{p}: M_k(P^{m-1}, P^n) \to M(E^{2m}, P^n) \). It follows that \( \iota^k_{m-1} \) is an orientable fibration with fibre

\[
F_m^k = \{ f \in M_k(P^m, P^n) | f \circ \iota_{m-1} = \tilde{g}_k \circ \iota_{m-1} \}
\]

homotopically equivalent to \( \Omega^{2m}P^n \). Here, and in the following, \( \tilde{g}_k: P^m \to P^m \subset P^n \) is the cellular map of degree \( k \) which in homogeneous coordinates is given by

\[
\tilde{g}_k[z_0 : z_1 : \cdots : z_m] = [z_0^k : z_1^k : \cdots : z_m^k].
\]

For later use, we now calculate the effect of the map \( \tilde{g}_k: F_m^1 \to F_m^k \) on the homotopy groups of the fibres.

**Lemma 2.1.** The induced homomorphism \( (\tilde{g}_k)_*: \pi_*(F_m^1, \tilde{g}_k) \to \pi_*(F_m^k, \tilde{g}_k) \) is multiplication by \( k^m \).

**Proof.** For any \( f \in \Omega^{2m}P^n \), let \( g_k + f \) denote the composite map

\[
P^m \stackrel{\nu}{\to} P^m \vee S^{2m} \stackrel{g_k \vee f}{\to} P^n \vee P^n \to P^n
\]

where \( \vee \) is the folding map and \( \nu \) is obtained by collapsing the boundary of an embedded \( 2m \)-disc \( D^{2m} \subset P^m \) to the base point. By choosing \( D^{2m} \) suitably, we get a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega^{2m}P^n & \xrightarrow{f_k} & \Omega^{2m}P^n \\
g_1 + \cdot & \downarrow & \downarrow g_k + \cdot \\
F_m^1 & \xrightarrow{\tilde{g}_k} & F_m^k
\end{array}
\]

in which the vertical maps are homotopy equivalences, cf. [5, p. 194], and \( f_k: S^{2m} = P^m/P^{m-1} \to S^{2m} = P^n/P^{m-1} \) induced by \( g_k \). Thus it suffices to show that \( f_k \) has degree \( k^m \). To that end, consider the commutative diagram of integral cohomology groups
where the vertical homomorphisms, induced by natural inclusions and projections, are isomorphisms.

Let \( c \in H^2(P^m; \mathbb{Z}) \) be a generator. Then

\[
g^*_k(c^m) = (g^*_k(c))^m = (kc)^m = k^m c^m
\]

since \( g^*_k \) has degree \( k \). As \( c^m \) generates \( H^2m(P^m) \), this shows that \( \deg f^*_k = k^m \). \( \square \)

Let \( E^k(P^m, P^n) \) denote the fibration

\[
e^k: M^k(P^m, P^n) \to F^n
\]

defined by evaluation at the base point \(* \in P^m \), i.e. \( e^k(f) = f(*) \) for \( f \in M^k(P^m, P^n) \). The fibre of \( E^k(P^m, P^n) \) is the space \( F^k(P^m, P^n) \) of based maps of degree \( k \) of \( P^m \) into \( P^n \). Also for these spaces of based maps, there are restriction fibrations

\[
i^{k}_{m-1}: F^k(P^m, P^n) \to F^k(P^{m-1}, P^n)
\]

with \( \Omega^{2m}P^n \) as fibre, \( m > 1 \). Using this, an inductive argument yields

**Lemma 2.2.** \( F^k(P^m, P^n) \) is \((2n - 2m)\)-connected and \( \pi_{2n-2m+1}(F^k(P^m, P^n)) \cong \mathbb{Z} \).

**Theorem 2.3.** The induced homomorphism \( e^*_k: H^r(P^n) \to H^r(M^k(P^m, P^n)) \) is an isomorphism for \( 0 \leq r \leq 2n - 2m \).

3. **Proof of Theorem.** First assume \( 1 < m < n \). Recall that

\[
i_k^{m-1}: M^k(P^m, P^n) \to M^k(P^{m-1}, P^n),
\]

as the pull-back of \( i: M(E^2m, P^n) \to M(E^2m, P^n) \), is an orientable fibration with fibre \( \Omega^{2m}P^n \). By the Freudenthal Suspension Theorem, \( \Omega^{2m}P^n \) is equivalent to \( S^{2n-2m+1} \) in dimensions \( < 4n - 4m + 1 \). Hence \( i_k^{m-1} \) has an associated Gysin sequence (integer coefficients) of the form

\[
0 \to H^{2n-2m+1}(M^k(P^m, P^n)) \to H^0(M^k(P^{m-1}, P^n))
\]

\[
\gamma^* \to H^{2n-2m+2}(M^k(P^{m-1}, P^n)) (i_{m-1}^{k-1})^* \to H^{2n-2m+2}(M^k(P^m, P^n)) \to 0.
\]

To get the trivial groups at the ends of this exact sequence we have used that \( H^{2n-2m+1}(M^k(P^{m-1}, P^n)) = H^1(M^k(P^{m-1}, P^n)) = 0 \) by Corollary 2.3. In this Gysin sequence, \( \gamma^k \) is cup product with the primary obstruction

\[
u^k \in H^{2n-2m+2}(M^k(P^{m-1}, P^n)) \cong \mathbb{Z}
\]
to constructing a cross-section of the fibration $i^k_{m-1}$. The Theorem will follow once we have computed $u^k$. In fact, we only need to compute $u^1$, for the map $g_k: (P^m, P^{m-1}) \rightarrow (P^m, P^{m-1})$ of degree $k$ induces a fibre map

$$M_1(P^m, P^n) \xrightarrow{\bar{g}_k} M_k(P^m, P^n) \downarrow i^k_{m-1} \downarrow i^k_{m-1}$$

$$M_1(P^{m-1}, P^n) \xrightarrow{\bar{g}_k} M_k(P^{m-1}, P^n)$$

of $i^k_{m-1}$ into $i^k_{m-1}$ and since

$$(\bar{g}_k)^*: H^{2n-2m+2}(M_k(P^{m-1}, P^n)) \rightarrow H^{2n-2m+2}(M_1(P^{m-1}, P^n))$$

is an isomorphism by Corollary 2.3, we deduce from Lemma 2.1 that $u^k = k^m u^1$. We have, therefore, reduced the Theorem to the following

**Lemma 3.1.** The formula $u^1 = \pm (n-m)$ holds for the primary obstruction $u^1 \in H^{2n-2m+2}(M_1(P^{m-1}, P^n)) \cong \mathbb{Z}$.

Before the proof of Lemma 3.1 we need a little preparation. With notation as in [5], let $\sigma_{n+1,m}$ be the sphere bundle

$$S^{2n-2m+1} \rightarrow U(n+1)/\Delta_{m+1} \times U(n-m) \rightarrow U(n+1)/\Delta_m \times U(n+1-m)$$

over a quotient $W_{n+1,m}/\Delta_m = U(n+1)/\Delta_m \times U(n+1-m)$ of the complex Stiefel manifold $W_{n+1,m} = U(n+1)/U(n+1-m)$ by an action of the group $\Delta_m = U(1)$. According to [5, Theorem 1.1], the primary obstruction $u^1$ agrees (up to sign) with the Euler class of $\sigma_{n+1,m}$.

Let $\eta: W_{n+1,m} \rightarrow W_{n+1,m}/\Delta_m$ be the canonical principal $U(1)$-bundle and $\xi: U(n+1)/\Delta_m \rightarrow W_{n+1,m}/\Delta_m$ the canonical principal $U(n+1-m)$-bundle over the base space $W_{n+1,m}/\Delta_m$ of $\sigma_{n+1,m}$. With the aid of the complex vector bundles $\eta[C]$ and $\xi[C^{n+1-m}]$ associated to $\eta$ and $\xi$, respectively, we may give an alternative description of $\sigma_{n+1,m}$.

**Lemma 3.2.** The sphere bundle $\sigma_{n+1,m}$ is fibre homotopically equivalent to the sphere bundle $S(\eta[C] \otimes \xi[C^{n+1-m}])$ of the tensor product of the conjugate bundle $\bar{\eta}[C]$ of $\eta[C]$ with $\xi[C^{n+1-m}]$.

Taking this for granted we return to the

**Proof of Lemma 3.1.** Let $c(\eta) = 1 + c_1(\eta)$ and $c(\xi) = 1 + \sum c_i(\xi)$ be the total Chern classes of $\eta$ and $\xi$. As $W_{n+1,m}$ is $(2n-2m+2)$-connected, the infinite cyclic group $H^2(W_{n+1,m}/\Delta_m; \mathbb{Z})$ is generated by $c_i(\eta)$ for $i \leq n+1-m$. Thus $c(\xi)$ can be expressed by the powers of $c_1(\eta)$. In fact,

$$c_i(\xi) = (-m)^i c_1(\eta)^i, \quad 1 \leq i \leq n+1-m,$$

for it is easily seen that the Whitney sum $\eta[C]^m \otimes \xi[C^{n+1-m}]$ is trivial.
Since \( u^1 \) is the Euler class of \( \pi_{n+1,m} \) we conclude, using Lemma 3.2, that up to sign
\[
  u^1 = c_{n+1-m}(\bar{\eta}[C] \otimes \xi[C^{n+1-m}]) = \sum_{i=0}^{n+1-m} c_1(\bar{\eta})^{n+1-m-i} \cup c_i(\xi)
\]
\[
  = \sum_{i=0}^{n+1-m} (-1)^i \binom{-m}{i} c_i(\eta)^{n+1-m} = \binom{n+1}{m} c_1(\eta)^{n+1-m}.
\]

**Proof of Lemma 3.2.** Let \( \Delta_m \times U(n + 1 - m) \) act on \( C \otimes C^{n+1-m} = C^{n+1} \) by \( (z, B) \cdot v = \bar{z} B v \) for \( z \in \Delta_m = S^1 \subset C, B \in U(n + 1 - m), \) and \( v \in C^{n+1-m} \). Then the stabilizer of the basis vector \( e_{m+1} \in \{0\} \times C^{n+1-m} \subset C^{n+1} \) is \( \Delta_{m+1} \times U(n-m) \), so since the diagonal map
\[
  D: U(n + 1) \to U(n + 1)/U(n + 1 - m) \times U(n + 1)/\Delta_m
\]
is a \( \Delta_m \times U(n + 1 - m) \)-map, it follows that
\[
  D \times e_{m+1}: U(n + 1) \to U(n + 1)/U(n + 1 - m) \times U(n + 1)/\Delta_m \times S^{2n-2m+1}
\]
induces a fibre homotopy equivalence
\[
  \pi_{n+1-m} \cong d^*(\eta \times \xi)[S^{2n-2m+1}]
\]
between \( \pi_{n+1,m} \) and the pull-back along the diagonal map \( d \) of \( W_{n+1,m}/\Delta_m \) of the fibre bundle \( (\eta \times \xi)[S^{2n-2m+1}] \) associated to the principal \( \Delta_m \times U(n + 1 - m) \)-bundle \( \xi \times \eta \) over \( W_{n+1,m}/\Delta_m \times W_{n+1,m}/\Delta_m \). But clearly,
\[
  d^*(\eta \times \xi)[S^{2n-2m+1}] = S(d^*(\eta \times \xi)[C^{n+1-m}]) = S(\bar{\eta}[C] \otimes \xi[C^{n+1-m}]).
\]

The Theorem has now been proved for \( 1 < m < n \). The case \( 1 < m \leq n \) is covered by [2, 4], but may also be obtained by some minor changes in the above proof. For \( m = n \), a slightly stronger statement holds.

**Proposition 3.3.** The fundamental group of \( M_k(P^n, P^n), n \geq 1, \) is cyclic of infinite order for \( k = 0 \) and cyclic of order \( (n+1)|k|^n \) for \( k \neq 0 \):
\[
  \pi_1(M_k(P^n, P^n)) \cong \mathbb{Z}/(n+1)|k|^n \mathbb{Z}.
\]

**Proof.** As the case \( n = 1 \) was handled in [4] we may assume that \( n > 1 \). Composition with \( g_k: (P^n, P^{n-1}) \to (P^n, P^{n-1}) \) defines a fibre map \( \tilde{g}_k \) of \( i_{n-1} \) into \( i_{n-1}^k \) which induces a map
\[
  \begin{array}{cccc}
    \pi_2(M_1(P^{n-1}, P^n)) & \xrightarrow{\partial_1} & \pi_1(F_n) & \xrightarrow{\partial_1} & \pi_1(M_1(P^n, P^n)) & \xrightarrow{m_2} & 0 \\
    m_2 \downarrow & & m_1 \downarrow & & \downarrow & & \downarrow \\
    \pi_2(M_k(P^{n-1}, P^n)) & \xrightarrow{\partial_2} & \pi_1(F_n^k) & \xrightarrow{\partial_1} & \pi_1(M_k(P^n, P^n)) & \xrightarrow{m_2} & 0
  \end{array}
\]

between the homotopy sequences. Since
\[
  (e_r)_*: \pi_2(M_k(P^{n-1}, P^n)) \to \pi_2(P^n), \quad r = 1, k,
\]
is an isomorphism by Lemma 2.2, it follows that \( m_2 \) is an isomorphism. As \( \pi_1(M_1(P^n, P^n)) = \mathbb{Z}_{n+1} \) by [4], \( \partial_1 \) must be multiplication by \( n+1 \). Using Lemma 2.1, we see that \( \partial_2 = \partial_2 \circ m_2 = m_1 \circ \partial_1 \) is multiplication by \( (n+1)|k|^n \). Hence
\[
  \pi_1(M_k(P^n, P^n)) \cong \pi_1(F_n^k)/\text{im} \partial_2 \cong \mathbb{Z}/(n+1)|k|^n \mathbb{Z}.
\]
\[
\]
References


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