PRIMITIVE OBSTRUCTIONS IN
THE COHOMOLOGY OF LOOPSPACES

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Abstract. Let X and X' be H-spaces. If f: \Omega X \to \Omega X' is an H-map then the
obstruction to f being a homotopy-commutative map is a subset \( \{ c_2(f) \} \subset [\Omega X \land \Omega X; \Omega^2 X'] \). In this paper we prove: If \( \alpha \) is in the image of the composition

\[ P_k \Omega X \land P_m \Omega X \to [\Omega X \land \Omega X; \Omega X'] \to [\Omega X \land \Omega X; \Omega^2 X'], \]

then \( \{ c_2(f) \} \) is in the image of the composition

\[ P_k \Omega X \land P_m \Omega X \to [\Omega X \land \Omega X; \Omega X'] \to [\Omega X \land \Omega X; \Omega^2 X']. \]

Consequently if \( \alpha \in H^n(\Omega X; \mathbb{Z}_p) \) is an \( A_3 \)-class in the sense of Stasheff then each
element of \( \{ c_2(f) \} \) is of the form \( \Sigma c'_i \otimes c''_i \) where the \( c'_i \) are primitive.

1. The purpose of this note is to develop a decomposition formula for a certain
obstruction class that occurs in the study of H-spaces. Let G and G' be associative
H-spaces and \( f: G \to G' \) be an H-map. For homotopy-commutative G and G' we
introduced in [6] the notion of \( f \) being a C_2-map. Specifically, if \( q \) and \( q' \) are
the commuting homotopies for G and G', respectively, and \( m \) is a homotopy from \( f(xy) \)
to \( f(x)f(y) \), then \( f \) is a C_2-map provided that there exists a secondary homotopy \( r: \]
\[ I^2 \times G^2 \to G' \] such that \( r(0, t, x, y) = f(q(t, x, y)) \), \( r(1, t, x, y) = q'(t, f(x), f(y)), \]
\( r(s, 0, x, y) = m(s, x, y) \), and \( r(s, 1, x, y) = m(s, y, x) \). The obstruction to
the existence of \( r \) is an element \( c_2(f) \) of \( [G \land G; \Omega G'] \), cf. [7]. Different choices of \( m \)
give us a set of obstructions \( \{ c_2(f) \} \subset [G \land G; \Omega G'] \).

The sets \( \{ c_2(f) \} \) have proved to be useful in the study of H-spaces, see for
example [7, 1]. We shall deal exclusively with the case \( G = \Omega X, G' = \Omega X' \), where X
and X' are H-spaces and \( q, q' \) are the usual commuting homotopies for the loop
multiplications. Now if \( G' = K(\mathbb{Z}_p, n) \), then

\[ \{ c_2(f) \} \subset [\Omega X \land \Omega X; \Omega K(\mathbb{Z}_p, n)] \cong H^{n-1}(\Omega X \land \Omega X; Z_p). \]

Zabrodsky proved in [7] that if \( \alpha \in H^n(\Omega X, Z_p) \) is a suspension element then the
elements of \( \{ c_2(f) \} \) may be written in the form \( \Sigma c'_i \otimes c''_i \) where the classes \( c'_i \) and
\( c''_i \) are also suspensions. One might hope that if \( \alpha \) were merely a primitive element
then the \( c'_i \) and \( c''_i \) might also turn out to be primitive. This is in general false. In
order to describe what is true in this case, we refer to Stasheff [3], for the definition of the projective spaces of $G$,

$$
\Sigma G = P_1G \subset P_2G \subset \cdots \subset \bigcup_k P_kG = P_\infty G \sim X.
$$

A map $f: G \to G'$ is called an $A_k$-map provided that $\Sigma f: \Sigma G \to \Sigma G'$ extends to a map of filtered spaces $\{P_i f\}: \{P_i G\} \to \{P_i G'\}$. The adjoint of the inclusion $i(1, k): \Sigma G \to P_kG$ is an $A_k$-map and the adjoint to $i(1, \infty)$ gives the homotopy equivalence of $G$ with $\Omega P_\infty G$. Under this equivalence the $A_\infty$-maps correspond to the loop maps. Furthermore, the $A_2$-maps are simply the $H$-maps. A class $\alpha \in [G; G']$ is represented by an $A_k$-map if and only if it is in the image of the composition $[P_kG; P_\infty G'] \to [\Sigma G; P_\infty G'] \isom [G; G']$. We shall prove

1.1. Theorem. If $G$ and $G'$ are loop spaces of $H$-spaces and $f: G \to G'$ is an $A_{k+m}$-map, then $\{c_2(f)\}$ is in the image of the composition

$$
[P_kG \wedge P_mG; P_\infty G'] \to [\Sigma G \wedge \Sigma G; P_\infty G'] \isom [G \wedge G; \Omega G'].
$$

Specializing to the case $G' = K(Z_p, n)$, Theorem 1.1 and the Künneth formula give us

1.2. Theorem. Let $X$ be an $H$-space. If $\alpha \in H^n(\Omega X; Z_p)$ is an $A_{k+m}$-class, then $\{c_2(\alpha)\}$ consists of elements of the form $c_2(\alpha) = \sum c_i' \otimes c_i''$, where the $c_i'$ (resp. $c_i''$) are $A_k$-classes (resp. $A_m$-classes).

(Since the $A_\infty$-classes are the suspension elements, we note that the case $k = m = \infty$ is the above-mentioned result of Zabrodsky.)

In particular, we see that if $\alpha$ is an $A_2$-class then we may write elements of $\{c_2(f)\}$ in the form $c_2(f) = \sum c_i' \otimes c_i''$, where the $c_i''$ are primitive. The import of Theorem 1.2 is thus that the types of elements that occur in the formula for $c_2(f)$ are considerably restricted. This fact is now being applied in the further investigation of the cohomology of finite $H$-spaces (cf. [2]), to extend results originally proved in [5].

2. The proof of Theorem 1.1 consists of the identification of $c_2(f)$ with another obstruction, $\theta(f)$, together with some routine diagram-chasing. Let us write $G = \Omega X$, $G' = \Omega X'$, for $H$-spaces $X$ and $X'$. The homotopy equivalence $X \sim P_\infty G$ induces a multiplication $\mu$ on $P_\infty G$ that may be taken to be filtration-preserving, i.e. $\mu(P_kG \times P_mG) \subset P_{k+m}G$ [4]. If $f: G \to G'$ is an $A_k$-map, $k \geq 2$, we have the (not necessarily commutative) diagram:

\[
\begin{array}{ccc}
\Sigma G \times \Sigma G & \xrightarrow{\mu} & P_2G \\
\Sigma f \times \Sigma f \downarrow & & \downarrow P_2f \\
\Sigma G' \times \Sigma G' & \xrightarrow{\mu'} & P_2G' \xrightarrow{i'(2, \infty)} P_\infty G'
\end{array}
\]

The obstruction $\theta(f)$ to a homotopy between $i'(2, \infty) \circ \mu' \circ (\Sigma f \times \Sigma f)$ and $i'(2, \infty) \circ P_2f \circ \mu$ is an element of $[\Sigma G \wedge \Sigma G; P_\infty G']$. The next proposition relates $\theta(f)$ to the question at hand.
2.1. **Proposition.** Under the isomorphism $[\Sigma G \wedge \Sigma G; P_\infty G] \cong [G \wedge G; \Omega G]', \theta(f)$ goes to $c_2(f)$.

**Proof.** This proposition is essentially a version of the adjoint relationship between Whitehead products and Samelson products. It follows from carefully depicting the maps and homotopies that are involved. □

Theorem 1.1 now follows from

2.2. **Proposition.** If $f: G \to G'$ is an $A_{k+m}$-map, then $\theta(f)$ is in the image of $(i(1, k) \wedge i(1, m))^*: [P_k G \wedge P_m G; P_\infty G'] \to [\Sigma G \wedge \Sigma G; P_\infty G]$.

**Proof.** By definition

$$\theta(f) = i'(2, \infty) \circ \mu' \circ (\Sigma f \times \Sigma f) - i'(2, \infty) \circ P_2 f \circ \mu.$$ 

By the commutative diagram

\[
\begin{array}{ccc}
\Sigma G \times \Sigma G & \xrightarrow{i(1,k) \times i(1,m)} & P_k G \times P_m G \\
\downarrow \Sigma f \times \Sigma f & & \downarrow P_k f \times P_m f \\
\Sigma G' \times \Sigma G' & \xrightarrow{i'(1,k) \times i'(1,m)} & P_k G' \times P_m G' \\
\downarrow \mu' & & \downarrow \mu' \\
P_2 G' & \xrightarrow{i'(2,k+m)} & P_{k+m} G' \\
\downarrow i'(2,\infty) & & \downarrow i'(k+m,\infty) \\
P_\infty G'
\end{array}
\]

we see that

$$i'(2, \infty) \circ \mu' \circ (\Sigma f \times \Sigma f) \sim i'(k+m, \infty) \circ \mu' \circ (P_k f \times P_m f) \circ (i(1, k) \times i(1, m)).$$

And, by the commutative diagram

\[
\begin{array}{ccc}
\Sigma G \times \Sigma G & \xrightarrow{i(1,k) \times i(1,m)} & P_k G \times P_m G \\
\downarrow \mu & & \downarrow \mu \\
P_2 G & \xrightarrow{i(2,k+m)} & P_{k+m} G \\
\downarrow P_2 f & & \downarrow P_{k+m} f \\
P_2 G' & \xrightarrow{i'(2,k+m)} & P_{k+m} G' \\
\downarrow i'(2,\infty) & & \downarrow i'(k+m,\infty) \\
P_\infty G'
\end{array}
\]

we obtain

$$i'(2, \infty) \circ P_2 f \circ \mu - i'(k+m) \circ P_{k+m} f \circ \mu \circ (i(1, k) \times i(1, m)).$$

So

$$\theta(f) = [i'(k+m, \infty) \circ \mu' \circ (P_k f \times P_m f) - i'(k+m, \infty) \circ P_{k+m} f \circ \mu \circ (i(1, k) \times 1(1, m))].$$
REFERENCES

2. , A seven-connected finite $H$-space is fourteen-connected (to appear).

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