

## STABILITY THEOREMS FOR CONFORMAL FOLIATIONS

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ABSTRACT. The global stability theorem of G. Reeb is no longer true if the codimension of the foliation is greater than one. However, in the presence of a complete transverse Riemannian structure, B. Reinhart obtained a global stability result. We prove global stability theorems for the much larger class of conformal foliations.

**1. Introduction.** An important problem in foliation theory is the study of the influence exerted by a compact leaf upon the global structure of a foliation. For certain classes of foliations, this influence is considerable.

**THEOREM (REEB STABILITY [7]).** *Let  $\mathcal{F}$  be a codimension one foliation of a compact connected manifold  $M$ . If  $\mathcal{F}$  has a compact leaf with finite fundamental group, then all the leaves of  $\mathcal{F}$  are compact with finite fundamental group.*

This theorem is false for foliations of codimension greater than one. However, in the presence of a certain transverse geometric structure one has the following global stability result.

**THEOREM (REINHART STABILITY [8]).** *Let  $\mathcal{F}$  be a complete Riemannian foliation of a connected manifold  $M$ . Then all the leaves of  $\mathcal{F}$  have the same universal covering space. In particular, if  $\mathcal{F}$  has a compact leaf with finite fundamental group, then all the leaves of  $\mathcal{F}$  are compact with finite fundamental group.*

Recall that a codimension  $q$  foliation  $\mathcal{F}$  of a manifold  $M$  is conformal if the frame bundle of its normal bundle admits a foliate reduction to the conformal group  $\text{CO}(q)$  or, equivalently, if its normal bundle admits a basic connection whose holonomy group is contained in  $\text{CO}(q)$  [11]. Equivalently,  $\mathcal{F}$  admits a transverse  $\text{CO}(q)$ -structure in the sense of [1]. In §2 we consider foliations which admit a transverse  $G$ -structure of finite type and completeness of such foliations and we prove

**THEOREM 1.** *Let  $\mathcal{F}$  be a complete conformal foliation of codimension  $q \geq 3$  of a connected manifold  $M$ . Then all the leaves of  $\mathcal{F}$  have the same universal covering space.*

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Stability theorems often involve some finiteness condition on the holonomy of a compact leaf. The local stability theorem of Reeb [7] is of this type and in [5] it is shown that if  $\mathcal{F}$  is a complete Riemannian foliation possessing a compact leaf with finite linear holonomy group, then all the leaves of  $\mathcal{F}$  are compact with finite linear holonomy group. If  $L$  is a leaf of a foliation, we denote by  $h^r(L, x)$  the infinitesimal holonomy group of order  $r$  of  $L$  at  $x \in L$ . Of course,  $h^1(L, x)$  is just the linear holonomy group of  $L$ . In §3 we prove

**THEOREM 2.** *Let  $\mathcal{F}$  be a complete conformal foliation of codimension  $q \geq 3$  of a connected manifold  $M$ . If  $\mathcal{F}$  has a compact leaf  $L_0$  with  $h^2(L_0, x)$  finite, then all the leaves of  $\mathcal{F}$  are compact and  $h^2(L, x)$  is finite for all  $L \in \mathcal{F}$ .*

**2. Prolongations of transverse  $G$ -structures.**

(2.1) **DEFINITION.** Let  $(M, \mathcal{F})$  be a foliated manifold,  $\text{codim } \mathcal{F} = q$ . A transverse parallelism on  $(M, \mathcal{F})$  is a collection of vector fields  $\{X_1, \dots, X_q\}$  on  $M$  which at each point  $x$  of  $M$  spans a subspace of  $T_x(M)$  complementary to the subspace tangent to  $\mathcal{F}$  and satisfies  $[Y, X_i]$  is tangent to  $\mathcal{F}$  whenever  $Y$  is tangent to  $\mathcal{F}$  for  $i = 1, \dots, q$ . A complete transverse parallelism on  $(M, \mathcal{F})$  is a transverse parallelism  $\{X_1, \dots, X_q\}$  on  $(M, \mathcal{F})$  such that each  $X_i$  is a complete vector field on  $M$ . We say that  $(M, \mathcal{F})$  is transversely parallelizable (respectively, completely transversely parallelizable) if it admits a transverse parallelism (respectively, complete transverse parallelism).

(2.2) **LEMMA [6].** *Let  $(M, \mathcal{F})$  be completely transversely parallelizable with  $M$  connected. Then the group  $\text{Aut}(M, \mathcal{F})$  of diffeomorphisms of  $M$  which preserve  $\mathcal{F}$  acts transitively on  $M$ .*

**PROOF.** We first remark that if  $L$  is a leaf of  $\mathcal{F}$ , then for any  $p, q \in L$  there exists  $\phi \in \text{Aut}(M, \mathcal{F})$  such that  $\phi(p) = q$ . Let  $X_1, \dots, X_q$  be a complete transverse parallelism on  $(M, \mathcal{F})$ . For  $i = 1, \dots, q$  let  $\phi^i: \mathbf{R} \times M \rightarrow M$  be the action of  $\mathbf{R}$  on  $M$  generated by  $X_i$ . Then  $\phi^i_t \in \text{Aut}(M, \mathcal{F})$  for all  $t \in \mathbf{R}$ ,  $i = 1, \dots, q$ . Let  $p \in M$ . Define  $\Phi: \mathbf{R}^q \rightarrow M$  by

$$\Phi(t_1, \dots, t_q) = \phi^1_{t_1} \circ \phi^2_{t_2} \circ \dots \circ \phi^q_{t_q}(p).$$

Let  $f: U \rightarrow \mathbf{R}^q$  be a submersion constant along the leaves of  $\mathcal{F}|_U$ , where  $U$  is a neighborhood of  $p \in M$ . Since  $\Phi_{*0}(\partial/\partial x^i|_0) = X_{i_p}$  for  $i = 1, \dots, q$ , it follows that  $(f \circ \Phi)_{*0}: T_0(\mathbf{R}^q) \rightarrow T_{f(p)}(\mathbf{R}^q)$  is an isomorphism. Hence there is a neighborhood  $V$  of 0 in  $\mathbf{R}^q$  and a neighborhood  $W$  of  $f(p)$  in  $\mathbf{R}^q$  such that  $\Phi(V) \subset U$  and  $f \circ \Phi: V \rightarrow W$  is a diffeomorphism. Hence  $\Phi: V \rightarrow M$  is an imbedded submanifold transverse to  $\mathcal{F}$ . Let  $L$  be any leaf in the saturation of  $\Phi(V)$ . Let  $q \in L \cap \Phi(V)$ . There is a point  $(t_1, \dots, t_q) \in V$  such that  $q = \Phi(t_1, \dots, t_q)$ . Thus  $q = \phi^1_{t_1} \circ \dots \circ \phi^q_{t_q}(p)$  and hence  $q$  is in the orbit of  $p$  under  $\text{Aut}(M, \mathcal{F})$ . By the earlier remark,  $L$  is contained in this orbit. Thus the saturation of  $\Phi(V)$  is contained in the orbit of  $p$ . Since this saturation is open, we have that the orbit of  $p$  is all of  $M$ .

We now construct prolongations of a transverse  $G$ -structure and, in particular, the first prolongation of a conformal foliation.

Let  $\mathcal{F}$  be a smooth codimension  $q$  foliation of the connected manifold  $M$ . Let  $E \subset T(M)$  be the tangent bundle of  $\mathcal{F}$  and let  $Q = T(M)/E$  be the normal bundle of  $\mathcal{F}$ . Let  $\rho: F(Q) \rightarrow M$  be the frame bundle of  $Q$ , a principal  $GL(q, \mathbf{R})$ -bundle. Let  $G$  be a Lie subgroup of  $GL(q, \mathbf{R})$ . Let  $P$  be a transverse  $G$ -structure on  $(M, \mathcal{F})$ . That is,  $P \subset F(Q)$  is a reduction to a principal  $G$ -bundle such that the natural parallelism along the leaves of  $\mathcal{F}$  carries elements of  $P$  to elements of  $P$  [1]. Note that  $\mathcal{F}$  is a  $G$ -foliation in the sense of [3]. There is a smooth  $q$ -dimensional manifold  $N$  and an  $N$ -cocycle  $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha, \beta \in A}$  defining  $\mathcal{F}$  such that the  $g_{\alpha\beta}$ 's are local automorphisms of a  $G$ -structure  $R \subset L(N)$  on  $N$ , where  $L(N)$  denotes the bundle of linear frames over  $N$ . Then  $\{(\rho^{-1}(U_\alpha), f_{\alpha_*}, g_{\alpha\beta_*})\}_{\alpha, \beta \in A}$  is an  $R$ -cocycle on  $P$  and hence defines a foliation  $\mathcal{F}_0$  of  $P$  with  $\dim \mathcal{F}_0 = \dim \mathcal{F}$ . If  $L_0 \in \mathcal{F}_0$ ; and  $L = \rho(L_0) \in \mathcal{F}$ , then  $\rho|_{L_0}: L_0 \rightarrow L$  is a regular covering.

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Let  $k \in \{0, 1, 2, \dots\}$ . The  $k$ th prolongation  $\mathfrak{g}_k$  of  $\mathfrak{g}$  is the space of symmetric multilinear mappings

$$t: \underbrace{\mathbf{R}^q \times \dots \times \mathbf{R}^q}_{(k+1)\text{-times}} \rightarrow \mathbf{R}^q$$

such that, for each fixed  $v_1, \dots, v_k \in \mathbf{R}^q$ , the linear transformation of  $\mathbf{R}^q$  given by  $v \rightarrow t(v, v_1, \dots, v_k)$  belongs to  $\mathfrak{g}$ . The  $k$ th prolongation  $G_k$  of  $G$  is the group of linear transformations  $t$  of  $V \oplus \mathfrak{g} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$  of the form

$$\begin{aligned} \tilde{t}(x) &= x \quad \text{for } x \in \mathfrak{g} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}, \\ \tilde{t}(v) &= v + t(\cdot, \dots, \cdot, v) \quad \text{for } v \in V \end{aligned}$$

where  $t \in \mathfrak{g}_k$  [4] (also cf. [10, 9]).

Let  $V = \mathbf{R}^q$ . As in [4], define  $\partial: \mathfrak{g} \otimes V^* \rightarrow V \otimes \Lambda^2 V^*$  by

$$(\partial f)(v_1, v_2) = -f(v_2)v_1 + f(v_1)v_2, \quad f \in \mathfrak{g} \otimes V^*, v_1, v_2 \in V.$$

Let  $C \subset V \otimes \Lambda^2 V^*$  be a subspace satisfying  $V \otimes \Lambda^2 V^* = \partial(\mathfrak{g} \otimes V^*) \oplus C$ . In general there is no natural choice of  $C$ .

Let  $\theta$  be the canonical form on  $P$ . It is the  $V$ -valued one-form on  $P$  defined by

$$\theta_u(Y) = u^{-1}(\pi \rho_{*u}(Y)), \quad u \in P, Y \in T_u(P),$$

where  $\pi: T(M) \rightarrow Q$  is the natural bundle projection and  $u: V \rightarrow Q_{\rho(u)}$  denotes the vector space isomorphism which sends the standard basis of  $V$  to the frame  $u$  of  $Q_{\rho(u)}$ . Let  $E_0 \subset T(P)$  be the subbundle tangent to  $\mathcal{F}_0$ . Then  $Q_0 = T(P)/E_0$  is the normal bundle of  $\mathcal{F}_0$ .

Let  $u \in P$ . Then  $\theta_u: T_u(P) \rightarrow V$  induces a map  $\theta_u: Q_{0_u} \rightarrow V$ . A  $q$ -dimensional subspace  $H$  of  $Q_{0_u}$  is called horizontal if  $\theta_u: H \rightarrow V$  is an isomorphism. Since  $i(X)d\theta = 0$  for all vector fields  $X$  tangent to  $\mathcal{F}_0$ , it follows that  $(d\theta)_u: T_u(P) \times T_u(P) \rightarrow V$  induces a skew-symmetric bilinear mapping  $(d\theta)_u: Q_{0_u} \times Q_{0_u} \rightarrow V$ . To each horizontal subspace  $H \subset Q_{0_u}$  we associate  $c(u, H) \in V \otimes \Lambda^2 V^*$  by restricting  $(d\theta)_u: Q_{0_u} \times Q_{0_u} \rightarrow V$  to  $H \times H$  and identifying  $H$  with  $V$  via  $\theta_u$ .

Let  $u \in P$  and let  $H \subset Q_{0_u}$  be a horizontal subspace. Define  $z_H: V \oplus \mathfrak{g} \rightarrow Q_{0_u}$  by

$$z_H(v \oplus A) = \theta_u^{-1}(v) + \pi(A_u^*),$$

where  $A^*$  is the fundamental vector field on  $P$  induced by  $A \in \mathfrak{g}$  and  $\pi: T_u(P) \rightarrow Q_{0_u}$  is the natural bundle projection. Then  $z_H$  is a linear isomorphism and hence may be regarded as a linear frame of  $Q_{0_u}$ . Let  $F(Q_0)$  be the frame bundle of  $Q_0$ . We define the first prolongation  $P_1$  of  $(P, \mathcal{F}_0)$  by

$$P_1 = \{ z_H \in F(Q_0): H \subset Q_{0_u} \text{ is horizontal, } u \in P, c(u, H) \in C \}.$$

Then, by an argument similar to that in [4],  $P_1$  is a transverse  $G_1$ -structure on  $(P, \mathcal{F}_0)$ . Note that  $P_1$  carries a foliation  $\mathcal{F}_1$  whose leaves are coverings of the leaves of  $\mathcal{F}_0$ . We define the  $k$ th prolongation  $(P_k, \mathcal{F}_k)$  of  $(P, \mathcal{F}_0)$  inductively by setting  $(P_k, \mathcal{F}_k) = ((P_{k-1})_1, (\mathcal{F}_{k-1})_1)$ , the first prolongation of  $(P_{k-1}, \mathcal{F}_{k-1})$ . Then  $(P_k, \mathcal{F}_k)$  is a transverse  $G_k$ -structure on  $(P_{k-1}, \mathcal{F}_{k-1})$ .

For the remainder of this section we assume that  $\mathcal{F}$  is a conformal foliation of  $M$  with  $q \geq 3$ . Thus

$$G = \text{CO}(q) = \{ A \in \text{GL}(q, \mathbf{R}): 'AA = cI, c \in \mathbf{R}, c > 0 \},$$

$$\mathfrak{g} = \text{co}(q) = \{ A \in \text{gl}(q, \mathbf{R}): 'A + A = cI, c \in \mathbf{R} \}.$$

Note that since the kernel of  $\partial$  is isomorphic to  $\mathfrak{g}_1$  and  $\dim \mathfrak{g}_1 = q$  [4], we have

$$\begin{aligned} \dim \text{image}(\partial) &= \dim \mathfrak{g} \otimes V^* - \dim \text{kernel}(\partial) \\ &= (q(q-1)/2 + 1)q - q = q^2(q-1)/2 = \dim V \otimes \Lambda^2 V^*, \end{aligned}$$

so  $\partial$  is onto. Thus the only choice of  $C$  is  $C = \{0\}$ , whence  $(P_1, \mathcal{F}_1)$  is canonically defined. Since  $\mathfrak{g}_2 = \{0\}$  [4], we have that  $(P_2, \mathcal{F}_2)$  is a transverse  $\{1\}$ -structure on  $(P_1, \mathcal{F}_1)$  so  $(P_1, \mathcal{F}_1)$  is transversely parallelizable. This is false for  $q = 1, 2$  since  $\text{co}(q)$  is of infinite type if  $q = 1, 2$ .

(2.3) DEFINITION. We say  $\mathcal{F}$  is a complete conformal foliation of  $M$  if  $(P_1, \mathcal{F}_1)$  is completely transversely parallelizable.

REMARK. If  $\mathcal{F}$  is a Riemannian foliation of  $M$  (i.e.,  $G = O(q)$ ), then  $\mathfrak{g}_1 = \{0\}$  and so  $(P, \mathcal{F}_0)$  is transversely parallelizable. If  $\mathcal{F}$  is a complete Riemannian foliation of  $M$ , then  $(P, \mathcal{F}_0)$  is completely transversely parallelizable.

Without loss of generality we may assume (by passing to a finite cover of  $M$  if necessary) that  $P$ , and hence  $P_1$ , is connected. If  $\mathcal{F}$  is a complete conformal foliation of  $M$  with  $q \geq 3$ , then  $\text{Aut}(P_1, \mathcal{F}_1)$  acts transitively on  $P_1$  by Lemma (2.2) and so all the leaves of  $\mathcal{F}_1$  are diffeomorphic. Since the leaves of  $\mathcal{F}_1$  are coverings of the leaves of  $\mathcal{F}$ , it follows that all the leaves of  $\mathcal{F}$  have the same universal covering space.

**3. Infinitesimal holonomy groups.** Let  $\mathcal{F}$  be a foliation of a manifold  $M$  and let  $L$  be a leaf of  $\mathcal{F}$ . Recall (see e.g. [2]) the definitions of the holonomy group and infinitesimal holonomy groups of  $L$ . Fix  $x_0 \in L$ . Let  $\sigma: [0, 1] \rightarrow L$  be a loop at  $x_0$ . Choose a covering of  $\sigma$  by a finite sequence of open sets  $U_0, U_1, \dots, U_{n-1}, U_n = U_0$  with  $x_0 \in U_0$ ,  $U_i \cap U_{i+1} \neq \emptyset$  for  $i = 0, 1, \dots, n-1$  such that, for each  $i = 0, 1, \dots, n$ ,  $f_i: U_i \rightarrow \mathbf{R}^q$  ( $q = \text{codim } \mathcal{F}$ ) is a submersion defining  $\mathcal{F}|U_i$ ,  $f_0 = f_n$ ,  $f_0(x_0) = 0$ . For each  $i = 0, 1, \dots, n-1$  there is a diffeomorphism  $g_{i+1,i}: f_i(U_i \cap U_{i+1}) \rightarrow f_{i+1}(U_i \cap U_{i+1})$  such that  $f_{i+1} = g_{i+1,i} \circ f_i$  on  $U_i \cap U_{i+1}$ . Then  $g_{n,n-1} \circ \dots \circ g_{1,0}$  is a local diffeomorphism defined in a neighborhood of 0 in  $\mathbf{R}^q$  fixing 0. Let  $h(\sigma)$  be its germ at 0. Since  $h(\sigma)$  depends only on the homotopy class

of  $\sigma$ , we obtain a homomorphism  $h: \pi_1(L, x_0) \rightarrow G(q)$ , well-defined up to conjugacy, where  $G(q)$  denotes the group of germs of local diffeomorphisms of  $\mathbf{R}^q$  fixing 0. The holonomy group of  $L$  based at  $x_0$  is defined by  $h(L, x_0) = \text{image}(h) \subset G(q)$ . A change of basepoint produces a conjugate subgroup. Let  $G^r(q)$  be the group of  $r$ -jets at 0  $\in \mathbf{R}^q$  and let  $\pi^r: G(q) \rightarrow G^r(q)$  be the natural projection. The infinitesimal holonomy group of order  $r$  of  $L$  at  $x_0$  is defined by  $h^r(L, x_0) = \text{image}(h^r) \subset G^r(q)$  where  $h^r = \pi^r \circ h$ .

The following construction gives a useful realization of  $h^r(L, x_0)$ . Let  $U$  and  $V$  be neighborhoods of 0 in  $\mathbf{R}^q$  and let  $f: U \rightarrow M, g: V \rightarrow M$  be smooth maps transverse to  $\mathcal{F}$  with  $f(0) = g(0) = x$ . Let  $W$  be a neighborhood of  $x$  and let  $F: W \rightarrow \mathbf{R}^q$  be a submersion defining  $\mathcal{F}|W$ . We say that  $f$  and  $g$  define the same transverse  $r$ -frame at  $x$  if  $F \circ f$  and  $F \circ g$  define the same  $r$ -frame at  $F(x)$ , i.e., if  $F \circ f$  and  $F \circ g$  have the same partial derivatives up to order  $r$  at 0. This definition is independent of the choice of the distinguished function  $F$ . The transverse  $r$ -frame determined by  $f$  is denoted by  $j_x^r(f)$ . Let  $P^r(M, \mathcal{F})$  be the set of transverse  $r$ -frames on  $M$ . Then  $\pi_r: P^r(M, \mathcal{F}) \rightarrow M, \pi_r(j_x^r(f)) = x$  is a principal bundle over  $M$  with group  $G^r(q)$ . The right action of  $G^r(q)$  on  $P^r(M, \mathcal{F})$  is given by  $j_x^r(f)j_0^r(g) = j_x^r(f \circ g)$  for  $j_x^r(f) \in P^r(M, \mathcal{F}), j_0^r(g) \in G^r(q)$ . Clearly  $P^1(M, \mathcal{F})$  is the bundle  $F(Q)$  of linear frames of the normal bundle of  $\mathcal{F}$  with group  $G^1(q) = \text{GL}(q, \mathbf{R})$ . Let  $F: W \rightarrow \mathbf{R}^q$  be a submersion defining  $\mathcal{F}|W$ . Then  $F$  induces a submersion  $F^{(r)}: P^r(M, \mathcal{F})|W \rightarrow P^r(\mathbf{R}^q)$  by  $F^{(r)}(j_x^r(f)) = j_{F(x)}^r(F \circ f)$ , where  $P^r(\mathbf{R}^q)$  denotes the bundle of  $r$ -frames of  $\mathbf{R}^q$ . Note that  $P^r(M, \mathcal{F})|W = F^{-1}(P^r(\mathbf{R}^q)|F(W))$ . Let  $\{(W_\alpha, F_\alpha, h_{\alpha\beta})\}_{\alpha, \beta \in A}$  be an  $\mathbf{R}^q$ -cocycle defining  $\mathcal{F}$ . Then  $\{(\pi_r^{-1}(W_\alpha), F_\alpha^{(r)}, h_{\alpha\beta}^{(r)})\}_{\alpha, \beta \in A}$  is a  $P^r(\mathbf{R}^q)$ -cocycle on  $P^r(M, \mathcal{F})$  and hence defines a foliation  $\mathcal{F}^r$  of  $P^r(M, \mathcal{F})$  with  $\dim \mathcal{F}^r = \dim \mathcal{F}$ . Let  $L \in \mathcal{F}$ . Then  $P^r(M, \mathcal{F})|L$  has a canonical flat connection whose holonomy group is isomorphic to  $h^r(L, x_0)$ . Let  $L^r \in \mathcal{F}^r$  be a leaf such that  $\pi_r(L^r) = L$ . Then  $\pi_r: L^r \rightarrow L$  is a regular covering whose group of covering transformations is isomorphic to  $h^r(L, x_0)$ .

(3.1) LEMMA. *The first prolongation of  $P^1(M, \mathcal{F})$  is  $P^2(M, \mathcal{F})$ ; i.e.,  $P^1(M, \mathcal{F})_1 = P^2(M, \mathcal{F})$ .*

PROOF. We first remark that the natural map  $P^2(M, \mathcal{F}) \rightarrow P^1(M, \mathcal{F})$  is a principal  $N_1^2(q)$ -bundle, where  $N_1^2(q)$  is the kernel of the natural map  $G^2(q) \rightarrow G^1(q)$ . Let  $j_x^2(f) \in P^2(M, \mathcal{F})$ . Then  $f: U \rightarrow M$  where  $U$  is a neighborhood of 0 in  $\mathbf{R}^q, f(0) = x$ , with  $f$  transverse to  $\mathcal{F}$ . Define  $\tilde{f}: P^1(\mathbf{R}^q)|U \rightarrow P^1(M, \mathcal{F})$  by  $\tilde{f}(j_y^1(g)) = j_{f(y)}^1(f \circ g)$ . Then  $\tilde{f}$  is transverse to  $\mathcal{F}^1$ . Let  $e = j_0^1(\text{Id}) \in P^1(\mathbf{R}^q)$  and let  $Q^1$  be the normal bundle of  $\mathcal{F}^1$ . Then  $\tilde{f}_{*e}: T_e(P^1(\mathbf{R}^q)) \rightarrow T_{j_x^1(f)}(P^1(M, \mathcal{F}))$  induces a linear isomorphism  $\tilde{f}_{*e}: T_e(P^1(\mathbf{R}^q)) \rightarrow Q_{j_x^1(f)}^1$ . This linear isomorphism determines a frame  $z$  of  $Q_{j_x^1(f)}^1$ .

Let  $\bar{H} \subset T(P^1(\mathbf{R}^q))$  be the horizontal distribution corresponding to the canonical flat linear connection on  $\mathbf{R}^q$  and let  $\bar{\theta}$  be the canonical  $V$ -valued one-form on  $P^1(\mathbf{R}^q)$ . Let  $H = \tilde{f}_{*e}(\bar{H}_e) \subset Q_{j_x^1(f)}^1$ . Then  $H$  is a horizontal subspace of  $Q_{j_x^1(f)}^1$ . Clearly  $z = z_H$ . Since  $d\bar{\theta}: \bar{H}_e \times \bar{H}_e \rightarrow V$  is zero (the canonical flat connection on  $\mathbf{R}^q$  has zero torsion) and since  $\tilde{f}^*(\theta) = \bar{\theta}$ , we have that  $d\theta: H \times H \rightarrow V$  is zero and so

$c(j_x^1(f), H) = 0$ . Since the only choice of  $C$  for  $G = GL(q, \mathbf{R})$  is  $C = \{0\}$ , we have that  $z = z_H \in P^1(M, \mathcal{F})_1$ . Thus the correspondence  $j_x^2(f) \rightarrow z$  defines a map  $P^2(M, \mathcal{F}) \rightarrow P^1(M, \mathcal{F})_1$ .

We show that this map is an isomorphism between the bundles  $P^2(M, \mathcal{F}) \rightarrow P^1(M, \mathcal{F})$  and  $P^1(M, \mathcal{F})_1 \rightarrow P^1(M, \mathcal{F})$  by exhibiting a compatible isomorphism between the groups  $N_1^2(q)$  and  $GL(q, \mathbf{R})_1$ . Let  $j_0^2(g) \in N_1^2(q)$ . There is a unique polynomial representation

$$g^i(y^1, \dots, y^q) = \sum a_j^i y^j + \sum a_{jk}^i y^j y^k, \quad i = 1, \dots, q,$$

where the coefficients  $a_{jk}^i$  are symmetric in the lower indices. Since  $j_0^1(g) = 0$ , we have  $a_j^i = 0$  for  $i, j = 1, \dots, q$ . Thus  $j_0^2(g)$  can be uniquely represented by the  $q$  symmetric  $q \times q$  matrices  $(a_{jk}^1), (a_{jk}^2), \dots, (a_{jk}^q)$ . Since  $GL(q, \mathbf{R})_1$  is naturally isomorphic to  $\mathfrak{gl}(q, \mathbf{R})_1 =$  the space of symmetric bilinear mappings  $V \times V \rightarrow V$ , the desired isomorphism of the two groups is immediate.

(3.2) LEMMA. *Let  $\mathcal{F}$  be a codimension  $q$  conformal foliation of  $M$  and let  $P$  be a transverse  $CO(q)$ -structure on  $(M, \mathcal{F})$ . Let  $p: P_1 \rightarrow M$  be the composition of the bundle projections  $P_1 \rightarrow P, P \rightarrow M$ . Let  $L$  be a leaf of  $\mathcal{F}$  and let  $L_1$  be a leaf of  $\mathcal{F}_1$  such that  $p(L_1) = L$ . Then  $p: L_1 \rightarrow L$  is a regular covering whose group of covering transformations is isomorphic to  $h^2(L, x_0)$ .*

PROOF. We define a bundle injection

$$\begin{array}{ccc} P_1 & \xrightarrow{\phi} & F(Q)_1 \\ \downarrow & & \downarrow \\ P & \xrightarrow{i} & F(Q) \end{array}$$

as follows. Let  $i = P \rightarrow F(Q)$  be the inclusion map. We map  $CO(q)_1$  into  $GL(q, \mathbf{R})_1$  by

$$CO(q)_1 \rightarrow co(q)_1 \xrightarrow{j} \mathfrak{gl}(q, \mathbf{R})_1 \rightarrow GL(q, \mathbf{R})_1$$

where  $j$  is the inclusion map and the other two arrows are the canonical isomorphisms. Let  $\nu(P, \mathcal{F}_0)$  (respectively,  $\nu(F(Q), \mathcal{F}_0)$ ) be the normal bundle of  $(P, \mathcal{F}_0)$  (respectively,  $(F(Q), \mathcal{F}_0)$ ). Let  $u \in P$ . Let  $H$  be a horizontal subspace of  $\nu(P, \mathcal{F}_0)_u$  such that  $c(u, H) = 0$  and let  $z_H$  be the corresponding element of  $P_1$ . Now  $i: P \rightarrow F(Q)$  induces  $i_*: \nu(P, \mathcal{F}_0)_u \rightarrow \nu(F(Q), \mathcal{F}_0)_u$ . Let  $H' = i_{*u}(H)$ . Then  $H'$  is a horizontal subspace of  $\nu(F(Q), \mathcal{F}_0)_u$  satisfying  $c(u, H') = 0$ . Let  $z_{H'}$  be the corresponding element of  $F(Q)_1$ . The correspondence  $z_H \rightarrow z_{H'}$  is then the desired map  $\phi: P_1 \rightarrow F(Q)_1$ .

Since  $F(Q)_1 = P^1(M, \mathcal{F})_1 = P^2(M, \mathcal{F})$  by Lemma (3.1), we obtain a bundle injection

$$\begin{array}{ccc} P_1 & \rightarrow & P^2(M, \mathcal{F}) \\ \downarrow & & \downarrow \\ P & \rightarrow & P^1(M, \mathcal{F}) \\ \searrow & & \swarrow \\ & M & \end{array}$$

Hence we may regard  $p: P_1 \rightarrow M$  as the restriction to  $P_1$  of  $\pi_2: P^2(M, \mathcal{F}) \rightarrow M$ , and so  $p: L_1 \rightarrow L$  is a regular covering whose group of covering transformations is isomorphic to  $h^2(L, x_0)$ , which completes the proof of the lemma.

Let  $\mathcal{F}$  be a complete conformal foliation of codimension  $q \geq 3$  of a connected manifold  $M$ . Let  $L_0$  be a compact leaf of  $\mathcal{F}$  with  $h^2(L_0, x)$  finite. Let  $L_1$  be a leaf of  $\mathcal{F}_1$  such that  $p(L_1) = L$ . By Lemma (3.2) we have that  $L_1$  is compact. Since  $(P_1, \mathcal{F}_1)$  is completely transversely parallelizable, it follows from Lemma (2.2) that all leaves of  $\mathcal{F}_1$  are compact. Also  $h^2(L, x)$  is finite for all  $L \in \mathcal{F}$  by Lemma (3.2) and so Theorem 2 is proved.

(3.3) COROLLARY. *Let  $\mathcal{F}$  be a complete conformal foliation of codimension  $q \geq 3$  of a connected manifold  $M$ . Then any two leaves of  $\mathcal{F}$  without holonomy are diffeomorphic.*

Since the germ at a point of a conformal transformation of a  $q$ -dimensional manifold with  $q \geq 3$  is determined by the 2-jet of the transformation at that point, we may rephrase Theorem 2 as follows.

THEOREM 2'. *Let  $\mathcal{F}$  be a complete conformal foliation of codimension  $q \geq 3$  of a connected manifold  $M$ . If  $\mathcal{F}$  has a compact leaf with finite holonomy group, then all the leaves of  $\mathcal{F}$  are compact with finite holonomy group.*

#### REFERENCES

1. L. Conlon, *Transversally parallelizable foliations of codimension two*, Trans. Amer. Math. Soc. **194** (1974), 79–102.
2. A. Haefliger, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1962), 367–397.
3. F. W. Kamber and P. Tondeur, *G-foliations and their characteristic classes*, Bull. Amer. Math. Soc. **84** (1978), 1086–1124.
4. S. Kobayashi, *Transformation groups in differential geometry*, Ergeb. Math. Grenzgeb., vol. 70, Springer-Verlag, Berlin, 1972.
5. P. Molino, *Propriétés cohomologiques et propriétés topologiques des feuilletages à connexion transverse projetable*, Topology **12** (1973), 317–325.
6. ———, *Géométrie globale des feuilletages riemanniens*, Proc. Nederl. Akad. Wetensch. A **85** (1982), 45–76.
7. G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Indust., no. 1183, Hermann, Paris, 1952.
8. B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. (2) **69** (1959), 119–132.
9. E. A. Ruh, *On the automorphism group of a G-structure*, Comment. Math. Helv. **39** (1964), 189–204.
10. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
11. I. Vaisman, *Conformal foliations*, Kodai Math. J. **2** (1979), 26–37.

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