ON THE RATIONALITY OF THE VARIETY OF SMOOTH RATIONAL SPACE CURVES WITH FIXED DEGREE AND NORMAL BUNDLE

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Abstract. Let \( S_{n,a} \) be the variety of smooth, rational curves of degree \( n \) in \( \mathbf{P}^3 \) whose normal bundle has a factor of degree \( 2n - 1 + a \) and a factor of degree \( 2n - 1 - a \). In this paper we prove that \( S_{n,a} \) is rational if \( n - a \) is even and \( a > 0 \).

We work over \( \mathbb{C} \). Let \( \hat{S}_{n,a} \subset \text{Hilb} \mathbf{P}^3 \) be the set of smooth, rational curves in \( \mathbf{P}^3 \) of degree \( n \) whose normal bundle splits with a summand of degree \( 2n - 1 - a \) and another of degree \( 2n - 1 + a \). Eisenbud and Van de Ven [1, 2] proved that for \( 0 \leq a \leq n - 4 \), \( \hat{S}_{n,a} \) is not empty, irreducible and of dimension \( 4n - 2a + 1 \) (if \( a > 0 \)). Let \( S_{n,a} \) be the set of embeddings \( f: \mathbf{P}^1 \to \mathbf{P}^3 \) with \( f(\mathbf{P}^1) \in \hat{S}_{n,a} \). They proved in [2] that \( S_{n,a} \) is irreducible, rational and, if \( a > 0 \), of dimension \( 4n - 2a + 4 \). \( \text{PGL}(2) = \text{Aut}(\mathbf{P}^1) \) acts naturally on \( S_{n,a} \) without fixed points. \( \hat{S}_{n,a} \) is the quotient of \( S_{n,a} \) by this action and the natural map \( S_{n,a} \to \hat{S}_{n,a} \) makes \( S_{n,a} \) a principal locally isotrivial bundle over \( \hat{S}_{n,a} \) with structural group \( \text{PGL}(2) \) (see Serre [6] for this notion).

In the introduction to [2] Eisenbud and Van de Ven raised the question of the rationality of \( \hat{S}_{n,a} \). Here we prove the following

**Theorem.** If \( a > 0 \) and \( n - a \) is even, then \( \hat{S}_{n,a} \) is rational.

The proof of this theorem uses only the construction in [2, §5], elementary properties of conic bundles (or \( \mathbf{P}^1 \)-bundles) with smooth fibers and smooth base, and the definition of stably rational varieties due to Kollar and Schreyer [4]. An irreducible variety \( V \) is said to be stably rational of level \( k \) if \( V \times \mathbf{P}^k \) is rational. For the elementary properties of conic bundles we need to see Serre [6]; we also found useful [3, 5].

We write \( \hat{S}_n \) for the variety of smooth, rational curves of degree \( n \) in \( \mathbf{P}^3 \) and \( S_n \) for the set of embeddings of degree \( n \) of \( \mathbf{P}^1 \) into \( \mathbf{P}^3 \). \( S_n \) is rational and \( S_n \to \hat{S}_n \) is a principal locally isotrivial bundle with structure group \( \text{PGL}(2) \). Since \( S_n \) (resp. \( S_{n,a} \)) is rational, if the natural map \( p: S_n \to \hat{S}_n \) (resp. \( S_{n,a} \to \hat{S}_{n,a} \)) has a rational section,
then \( \hat{S}_n \) (resp. \( \hat{S}_{n,a} \)) is stably rational of level 3. The rationality of \( S_{n,a} \) was proved in [2, p. 97].

**Lemma 1.** Assume \( n \) odd. Then for every \( x \in \hat{S}_n \), there exists a rational section of \( p \) defined at \( x \).

**Proof.** Since \( \hat{S}_n \) is contained in \( \text{Hilb} \, P_3 \), we have a universal curve \( C \to \hat{S}_n \) with an inclusion \( i: C \to \hat{S}_n \times P_3 \) over \( \hat{S}_n \). \( C \) is a conic bundle with a smooth base. Since \( n \) is odd, this conic bundle is locally trivial in the Zariski topology [2]. Thus there is a neighborhood \( U \) of \( x \) and an \( U \)-isomorphism \( h: U \times P_1 \to C \). The map \( i \circ h \) gives the section of \( p \) defined on \( U \). □

We write \( R_n \) for the set of maps of degree \( n \) of \( P_1 \) into \( P_3 \). Again \( \text{PGL}(2) \) acts on \( R_n \) and we write \( \hat{R}_n \) for its quotient. Since we are interested only at birational geometry, there is no problem here; we can substitute \( R_n \) with \( S_n \) if we want. In [2] a key point was the map \( G: S_{n,a} \to \hat{R}_{n-a-1} \) constructed in the following way. Fix \( f \in S_{n,a} \).

\[
N_f := f^* \left( N_{f(P_1/P_3)} \right) \cong \mathcal{O}_{P_1}(2n - 1 - a) \oplus \mathcal{O}_{P_1}(2n - 1 + a)
\]

is a quotient of \( f^*(TP_3) \). Thus the subline bundle \( \mathcal{O}_{P_1}(2n - 1 + a) \) defines a rank-2 subbundle \( V_f \) of \( f^*(TP_3) \). The map \( G(f): P_1 \to \hat{R}_n \) is constructed by taking for \( G(f)(t) \) the plane in \( P_3 \) which is determined by \( V_{f,t} \subset TP_{3,f(t)} \). Note that the map \( G \) descends to a map \( \hat{G}: \hat{S}_{n,a} \to \hat{R}_{n-a-1} \) such that, for \( 0 < a \leq n - 4 \) we have the following commutative diagram:

\[
\begin{array}{ccc}
S_{n,a} & \overset{G}{\longrightarrow} & R_{n-a-1} \\
\downarrow q & & \downarrow g \\
\hat{S}_{n,a} & \overset{\hat{G}}{\longrightarrow} & \hat{R}_{n-a-1} \\
\end{array}
\]

Eisenbud and Van de Ven [2, p. 97] proved that \( G \) is birationally the projection of a product with fiber rational of dimension \( 2a + 5 \). If \( n - a \) is even, by Lemma 1 \( g \) has a rational section. Thus \( \hat{R}_{n-a-1} \) is stably rational of level 3, \( \hat{G} \) is birationally a product with fiber \( P_{2a+5} \) and \( \hat{S}_{n,a} \) is rational. This concludes the proof of Theorem 1.

If \( n - a \) is odd, \( a > 0 \), we do not know very much. A trick easily gives the following

**Proposition 1.** Assume \( a > 0 \). Then \( \hat{S}_{n,a} \) is covered by rational subvarieties of codimension 2.

**Proof.** Fix a point \( O \in P_1 \) and a point \( P \) in \( P_3 \). Let \( A_n \) be the set of embeddings \( f \) of \( P_1 \) into \( P_3 \) with \( f(O) = P \) and \( \text{deg}(f(P_1)) = n \). \( A_n \) is rational. The affine group of projective transformations of \( P_1 \) fixing \( O \) acts on \( A_n \) and let \( \hat{A}_n \subset \text{Hilb} \, P_3 \) be the quotient. \( A_n \) is the subset of \( S_n \) formed by curves through \( P \). The map \( A_n \to \hat{A}_n \) has always a rational section. This follows from the speciality of the affine group [3, Lemme 2.3]. Alternatively the restriction to \( \hat{A}_n \) of the conic bundle of Lemma 1 comes from a vector bundle since the point \( P \) defines a line bundle on \( p^{-1}(\hat{A}_n) \) with degree one on every fiber.
Thus $\tilde{A}_{n-a-1}$ is stably rational of level 2 and $\tilde{G}^{-1}(\tilde{A}_{n-a-1})$ has a rational section. Thus $\tilde{G}^{-1}(\tilde{A}_{n-a-1})$ is a rational subvariety of codimension 2 of $\tilde{S}_{n,a}$. □

For $a = 0$ the same method gives only that $\tilde{S}_n$ is covered by codimension 2 subvarieties which are stably rational of level 2.

REFERENCES

4. J. Kollar and F. O. Schreyer, The moduli of curves is stably rational for $g \leq 6$ (preprint). 

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