MULTI-DIMENSIONAL ANALYTIC STRUCTURE
AND SHILOV BOUNDARIES

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Abstract. We give a condition under which multi-dimensional analytic structure can be introduced into the maximal ideal space of a uniform algebra.

Introduction. Let \( A \) be a uniform algebra on a compact Hausdorff space \( X \) and \( M \) the maximal ideal space of \( A \). Various authors have showed that one-dimensional or multi-dimensional analytic structure can be introduced into \( M \) provided that there exists a suitable subset \( G \) of \( \mathbb{C} \), or \( \mathbb{C}' \) respectively, with “the finite fibre property”. Thus, the classic theorem on the subject by E. Bishop [5] states

**Theorem 1.** For \( f \in A \), define \( f^{-1}(\lambda) = \{ x \in M : f(x) = \lambda \} \). Let \( W \) be a component of \( C \setminus f(X) \). Suppose that there exists a subset \( G \) of \( W \) such that:

1. \( G \) has positive two-dimensional Lebesgue measure.
2. For each \( \lambda \in G \), the cardinality of \( \{ f^{-1}(\lambda) \} \), is finite.

Then, there is an integer \( n \) such that for every \( \lambda \in W \), \( \# f^{-1}(\lambda) \leq n \). Furthermore, \( f^{-1}(W) \) can be given the structure of a one-dimensional complex analytic space such that each \( g \in A \) is holomorphic on this space.

B. Aupetit and J. Wermer [1] showed that the hypothesis on \( G \) of “positive measure” can be replaced by “positive exterior logarithmic capacity”, and no weaker condition will suffice.

Also generalizing Bishop’s result, R. Basener [3] and N. Sibony [9] independently formulated a condition for the existence of an \( n \)-dimensional analytic structure as follows: Let \( A' = \{ (f_1, \ldots, f_n) | f_1, \ldots, f_n \in A \} \), so that each \( F \in A' \) maps \( M \) to \( \mathbb{C}' \). Let \( V(F) = \{ x \in M : F(x) = (0, \ldots, 0) \} \). The \( n \)th Shilov boundary \( \partial_n A \) is defined by \( \partial_n A = \text{closure} \left[ \bigcup_{F \in A'} \partial_0 A_{V(F)} \right] \), \( \partial_0 A \) is the usual Shilov boundary.

**Theorem 2.** Fix \( F \in A' \). Let \( W \) be a component of \( F(M) \setminus F(\partial_{n-1} A) \). Suppose there exists \( G \subseteq W \) such that:

1. \( M_{2n}(G) > 0 \) (\( M_{2n} \) is the Lebesgue measure in \( \mathbb{C}' \)).
2. For each \( \lambda \in G \), \( \# F^{-1}(\lambda) \) is finite.
Then, there exists a positive integer k such that for all \( \lambda \in W \), \( \#F^{-1}(\lambda) \) is at most k. Moreover, \( S = (F^{-1}(W), F, W) \) is a branched analytic cover; consequently \( F^{-1}(W) \) is an analytic space and for every \( f \in A, f \) is holomorphic on \( F^{-1}(W) \).

When \( n = 1 \), this is Bishop’s theorem.

Recently B. Aupetit [2] improved Theorem 2 by replacing (1) with the condition that \( G \) is not pluri-polar. (\( G \) is not pluri-polar if there is no plurisubharmonic function \( \phi \) on \( C^n \) such that \( G \subset \{ \lambda \in C^n | \phi(\lambda) = -\infty \}.\)

Aupetit’s proof of the above requires that \( G \) is contained in \( F(M) \setminus F(\partial_{n-1}A) \). By the definition of the \( n \)th Shilov boundary, we have \( \partial_0 A \subseteq \partial_0 A_1 \subseteq \cdots \subseteq M \). Since a component of \( F(M) \setminus F(\partial_{n-1}A) \) is open in \( C^n \) [3, Lemma 2], it is contained in the interior of \( F(M) \setminus F(\partial_0 A) \). In this paper we formulate a condition for an \( n \)-dimensional analytic structure assuming that \( G \) is a non-pluri-polar set contained in the interior of \( F(M) \setminus F(\partial_0 A) \). The hypothesis of “non-pluri-polar set” on \( G \) is then replaced by a more general “uniqueness set”. (Let \( \Omega \) be a region in \( C^n \). \( G \subseteq \overline{\Omega} \) is a uniqueness set (for \( \Omega \)) if every plurisubharmonic function on \( \Omega \) that converges to \(-\infty\) at every point of \( G \) is identically equal to \(-\infty\) on \( \Omega \).)

Our main results are stated in Theorems 3 and 4. We make essential use of the plurisubharmonicity proof for a certain class of functions associated with a uniform algebra, which the author developed in [7] and extended in [8]. We give an example covered by Theorem 4 of this paper but not by the Basener-Sibony-Aupetit Theorem.

Example. Let \( \Delta^2 \) be the open unit bidisc in \( C^2 \) and \( A = A(\Delta^2) \). Then \( \partial_0 A = \{(z, w) \in \Delta^2: |z| = 1, |w| = 1\}; \partial_1 A = \{(z, w) \in \Delta^2: |z| = 1 \text{ or } |w| = 1\}; \) and \( \partial_2 A = \Delta^2 = M \). The set \( G = \{(z, w): |z| < 1, |w| = 1\} \) is a uniqueness set for \( \Delta^2 \) and its 4-dimensional Lebesgue measure is zero. If we take \( F \) to be the map \((z, w)\) then \( G \) is contained in \( F(M) \setminus F(\partial_0 A) \) but not in \( F(M) \setminus F(\partial_{n-1} A) \).

Theorem 4 can be readily extended to the case where \( \# F^{-1}(\lambda) \) is assumed to be countable on \( G \) in view of Basener’s Theorem [4].

We introduce some definitions and notations. Fix \( F \in A^n \) and denote by \( \mathcal{M} \) the subset of the Cartesian product of \( n \)-copies of \( M \) consisting of the points \( m = (m_1, \ldots, m_n) \) such that \( F(m_1) = \cdots = F(m_n) \). Define the projection \( \pi \) on \( \mathcal{M} \) by \( \pi(m) = F(m_1) \). Let \( \mathcal{A} \) be the uniform algebra on \( \mathcal{M} \) generated by the functions of the form \( \theta \to g_1(\theta_1) \cdots g_n(\theta_n), g_i \in A, i = 1, 2, \ldots, n \). The maximal ideal space of \( \mathcal{A} \) is \( \mathcal{M} \).

Lemma 1. Let \( \Omega \) be a component of the interior of \( F(M) \setminus F(\partial_0 A) \) in \( C^n \). For each \( \tau \in \mathcal{A} \), the function \( \phi \), defined by

\[
\phi(\lambda) = \log \left\{ \max_{\theta \in \pi^{-1}(\lambda)} |\tau(\theta)| \right\},
\]

is plurisubharmonic in \( \Omega \).

Proof. The upper semicontinuity of \( \phi \) is proved by a standard method (see [11, p. 139]). We must show that if \( L \) is a complex line contained in \( \Omega \) the restriction of \( \phi \) to \( L \) is subharmonic. Let \( D \) be a disc contained in \( L \). For some \( \alpha_{jk} \) and \( \gamma_k \) in \( C \),

\[
L = \bigcap_{k=1}^{n-1} \left( \lambda_1, \ldots, \lambda_n \right) \in \Omega \left\{ \sum_{j=1}^{n} \alpha_{jk} \lambda_j = \gamma_k \right\}.
\]
Put

\[ V = \left\{ m \in M \mid \sum_{j=1}^{n} \alpha_{jk} f_j(m) = \gamma_k, k = 1, \ldots, n - 1 \right\}. \]

By \( \mathcal{A}_V \) we mean the restriction algebra, \( \mathcal{A}_V = \{ f \in C(V) | f \) is the uniform limit of functions in \( \mathcal{A} \} \). The maximal ideal space of \( \mathcal{A}_V \) is \( V \). Choose a polynomial \( P \) with \( \phi \leq \text{Re} P \) on \( bD \). Then, for each \( \xi \) in \( bD \), \( \max_{\pi^{-1}(\xi)} |\tau| \cdot |e^{-P(\xi)}| \leq 1 \). We must show that \( \phi \leq \text{Re} P \) on \( \overline{D} \).

Fix \( z \in D \). There exists \( \theta \in \pi^{-1}(z) \) such that \( |\tau(\theta)| = \max_{\pi^{-1}(z)} |\tau| \). The function \( \theta \to \tau(\theta) \cdot e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))} \), restricted to \( V \), is in \( \mathcal{A}_V \). We shall show in Lemma 2 that \( \pi^{-1}(D) \subseteq V \setminus \partial_0[\mathcal{A}_V] \). Assuming that Lemma 2 is true, by the local maximum modulus principle of \( \mathcal{A}_V \) applied to \( \pi^{-1}(D) \),

\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))}| \leq |\tau(\alpha)| \cdot |e^{-P(f_1(\alpha_1), \ldots, f_n(\alpha_n))}| \]

for some \( \alpha = (\alpha_1, \ldots, \alpha_n) \in b[\pi^{-1}(D)] \subseteq \pi^{-1}(bD) \). Hence

\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))}| \leq 1. \]

Thus, for an arbitrary \( z \in \overline{D} \),

\[ \phi(z) = \log \max \{ |\tau(\theta)| : \theta \in \pi^{-1}(z) \} \leq \text{Re} P(z). \]

This proves Lemma 1 assuming that \( \pi^{-1}(D) \subseteq V \setminus \partial_0[\mathcal{A}_V] \), which is to be proven. Next we deduce an important consequence of Lemma 1.

**Corollary 1.1.** Fix \( g \in A, F \in A^n \). For each \( k \in \mathbb{N} \),

\[ \psi_{k,g}(\lambda) = \log \max \left\{ \prod_{1 \leq i, j \leq k} |g(\theta_i) - g(\theta_j)| : \theta_i, \theta_j \in F^{-1}(\lambda) \right\} \]

is plurisubharmonic on \( \Omega \).

**Proof.** \( \prod_{1 \leq i, j \leq k} (g(\theta_i) - g(\theta_j)) \in \mathcal{A} \). If \( \theta_1, \ldots, \theta_k \in F^{-1}(\lambda) \), then \( (\theta_1, \ldots, \theta_k) \in \pi^{-1}(\lambda) \). The plurisubharmonicity of \( \psi_{k,g} \) follows from Lemma 1.

**Theorem 3.** Fix \( g \in A \) and \( F \in A^n \). Let \( \Omega \) be a component in the interior of \( F(M) \setminus F(\partial_0 A) \) in \( \mathbb{C}^n \). Suppose that there exists a subset \( G \) of \( \Omega \) with the following properties:

(i) \( G \) is not pluri-polar.

(ii) For every \( \lambda \in G \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is finite.

Then, there exists \( k \in \mathbb{N} \) such that for each \( \lambda \in \Omega \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is at most \( k \).

**Proof.** The condition (ii) implies that \( G = \bigcup_{i \in \mathbb{N}} G_i \), where \( G_i = \{ \lambda \in G | g \) assumes \( i \) values on \( F^{-1}(\lambda) \} \). For some \( k \in \mathbb{N} \), \( G_k \) is non-pluri-polar. Since for each \( \lambda \in G_k \), \( g \) assume \( k \) values on \( F^{-1}(\lambda) \),

\[ \max_{\theta_i, \theta_j \in F^{-1}(\lambda)} \prod_{1 \leq i, j \leq k+1} |g(\theta_i) - g(\theta_j)| = 0 \]

on \( G_k \). Hence, \( \psi_{k+1,g} \equiv -\infty \) on \( \Omega \). This implies that \( g \) assumes at most \( k \) values on \( F^{-1}(\lambda) \) for each \( \lambda \in \Omega \). Take \( k \) to be the largest integer such that \( \Omega_k \cap \Omega \neq \emptyset \). Thus, \( \Omega = \bigcup_{i=1}^{k} \Omega_i \).
We shall now prove the hypothesis used in Lemma 1.

**Lemma 2.** Use notations as in Lemma 1.

\[ \pi^{-1}(D) \subseteq V \setminus \partial_0 [\mathcal{U}_V]. \]

**Proof.** Let \( s \in \pi^{-1}(D). \) Then \( \pi(s) = F(s_1) = (f_1(s_1), \ldots, f_n(s_1)) \in D \subseteq L. \) Therefore, \( s \in V. \) We need to show \( s \not\in \partial_0 [\mathcal{U}_V]. \) Put \( a = \pi(s). \) Let \( U \) be an open disc contained in \( L \) and centered at \( a. \) Let \( bU \) be the boundary of \( U \) in the topology of \( L. \) For each function \( H \) in \( \mathcal{U}_V \) we shall construct a bounded analytic function \( \gamma \) on \( U \) satisfying \( H(s) = \gamma(a) \) and \( |\gamma(a)| \leq \max_{\pi^{-1}(bU)} |H|. \) We may assume without loss of generality that \( H \) takes the form,

\[ (1) \quad H(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} h_{ij}(\theta_j); \quad (\theta_1, \ldots, \theta_n) \in V, l \in N, h_{ij} \in A. \]

Let \( U_j \) be the projection of \( U \) on the \( j \)th coordinate axis. \( F^{-1}(\overline{U}) = \bigcap_{j=1}^{l} \{ f_j^{-1}(\overline{U}_j) \} \) and it is \( \mathcal{A} \)-convex. Moreover, \( F^{-1}(\overline{U}) \cap \partial_0 A = \emptyset. \) Let \( \tilde{f}_j \) be the restriction of \( f_j \) to \( F^{-1}(\overline{U}). \) By the local maximum modulus principle applied to \( A \) with respect to \( F^{-1}(\overline{U}), \partial_0 AF^{-1}(\overline{U}) \subseteq \tilde{f}_i^{-1}(bU). \) Let \( \mu_j \) be a representing measure for \( s_j \) concentrated on \( \partial_0 AF^{-1}(\overline{U}). \) For each \( h_{ij} \) in (1), define

\[ \eta_{ij}(z) = \int_{\tilde{f}_j^{-1}(bU_j)} \frac{f_j - a_j}{f_j - z} h_{ij} d\mu_j. \]

**Assertion.** \( \eta_{ij} \) has the following properties:

(i) \( \eta_{ij} \) is bounded and analytic on \( U_j. \)
(ii) If \( \xi_j \in bU_j \) and a nontangential limit \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z) \) exists, then

\[ |\eta_{ij}(\xi_j)| \leq \max_{f_j^{-1}(\xi)} |h_{ij}|. \]

(iii) \( \eta_{ij}(a_j) = h_{ij}(s_j). \)

The proof for the assertion is the same as that used by Seničkin in [10, Lemma 7].

Using the functions, \( \eta_{ij} \) defined above, form a bounded analytic function \( \gamma \) on \( U \) as

\[ \gamma(z_1, \ldots, z_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} \eta_{ij}(z_j). \]

Choose \( \xi_j \in bU_j \) so that all the nontangential limits \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z), z \in U_j, \)

\( 1 \leq i \leq l, \) exist, and \( (\xi_1, \ldots, \xi_n) \in bU. \) Put

\[ \gamma(\xi_1, \ldots, \xi_n) = \lim_{(z_1, \ldots, z_n) \to (\xi_1, \ldots, \xi_n)} \gamma(z_1, \ldots, z_n). \]

Using essentially the same proof as in [8, Lemma 2] we obtain

\[ |\gamma(\xi_1, \ldots, \xi_n)| \leq \max_{\pi^{-1}(bU)} |H|. \]

This is true for almost all points in \( bU \) and \( \gamma \) is analytic. So,

\[ |\gamma(a_1, \ldots, a_n)| \leq \max_{\pi^{-1}(bU)} |H|. \]
Hence

\[ |H(s)| \leq \max_{\pi^{-1}(bU)} |H|. \]

Since \( s \) and \( H \) were chosen arbitrarily and \( \partial_0[\mathbb{A}_\nu] \) is the closure of the generalized peak points of \( \mathbb{A}_\nu \), this yields the desired conclusion

\[ \partial_0[\mathbb{A}_\nu] \cap \pi^{-1}(D) = \emptyset. \]

Theorem 3 can be extended, using the same proof, as follows.

**Definition.** Let \( \Omega \) be a region on \( \mathbb{C}^n \). We say \( G \subseteq \Omega \) is a set of uniqueness for \( \Omega \) if every plurisubharmonic function defined on \( \Omega \) that converges to \(-\infty\) at every point of \( G \) is identically equal to \(-\infty\) on \( \Omega \).

**Corollary 3.1.** Let \( \Omega \) be a component of the interior of \( F(M) \setminus F(\partial_0 A) \) and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \subseteq \overline{\Omega} \) satisfying

(i) \( G \) is a set of uniqueness for \( \Omega \).

(ii) For every \( \lambda \in G \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is finite.

Then, there exists \( k \in \mathbb{N} \) such that for each \( \lambda \) in \( \Omega \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is at most \( k \).

The following lemma is a special case of Theorem 4.

**Lemma 3.** Let \( A, M, \) and \( F \) be as before. Suppose that \( W \) is a component of \( F(M) \setminus F(\partial_{n-1} A) \), and suppose \( g \in A \) is constant on \( F^{-1}(\lambda) \) for every \( \lambda \in W \). Then, \( g \circ F^{-1} \) is analytic on \( W \).

**Proof.** We show \( g \circ F^{-1} \) is analytic in each variable. Let \( a \in W \) and \( \Delta''(a, r) \) be an open polydisc about \( a \); \( \Delta''(a, r) = \prod_{i=1}^n \Delta_i \), \( \Delta_i = \Delta(a_i, r_i) \). Put \( \Delta_i' = \{(a_1, \ldots, z_i, \ldots, a_n) | z_i \in \Delta_i \} \). Note that \( F^{-1}(\Delta_i') = f^{-1}_i(\Delta_i) \cap \Gamma \), where \( \Gamma \) is the set of zeros of \( n - 1 \) functions in \( A \). For simplicity of notation denote by \( f \) the restriction of \( f_i \) to \( \Gamma \). Thus \( F^{-1}(\Delta_i') = f^{-1}_i(\Delta_i) \), and we shall show that \( g \circ f^{-1} \) is analytic on \( \Delta_i' \).

Consider the algebra \( A_\Gamma \). By the definition of \( \partial_{n-1} A \), we have \( \partial_0 A_\Gamma \subseteq \partial_{n-1} A \), and \( \Delta_i \cap f(\partial_0 A_\Gamma) = \emptyset \) since \( \Delta(a, r) \cap F(\partial_{n-1} A) = \emptyset \). Note that \( \partial_0 A_{f^{-1}(\Delta_i)} \subseteq f^{-1}(b\Delta_i) \), by the local maximum modulus principle of \( A_\Gamma \) applied to \( f^{-1}(\Delta_i) \).

Let \( \mu_i \) be the representing measure for some \( m_i \in f^{-1}(a_i) \) supported on \( \partial_0 A_{f^{-1}(\Delta_i)} \), and \( \nu_i \) the projection of \( \mu_i \) on \( b\Delta_i \), which is the normalized Lebesgue measure on \( b\Delta_i \).

\[
\int_{b\Delta_i} g \circ f^{-1} \, d\nu_i = \int_{f^{-1}(b\Delta_i)} gd\mu_i = g(m_i) = g \circ f^{-1}(a_i).
\]

Thus, \( g \circ f^{-1} \) is a complex harmonic function.

\[
\int_{b\Delta_i} (z-a_i)^n \cdot g \circ f^{-1} \, d\nu_i = \int_{f^{-1}(b\Delta_i)} (f-a_i)^n \cdot gd\mu_i = 0 \quad (n \in \mathbb{N}).
\]

This shows that \( g \circ f^{-1} \) is holomorphic on \( \Delta_i \), and hence, \( g \circ F^{-1} \) on \( \Delta_i' \).

**Theorem 4.** Fix \( F \in A^n \) and \( g \in A \). Let \( \Omega \) be a component contained in the interior of \( F(M) \setminus F(\partial_0 A) \), and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \subseteq \overline{\Omega} \) such that:

(i) \( G \) is a set of uniqueness for \( \Omega \).
(ii) For every $\lambda \in G$, $\# \{ g \circ F^{-1}(\lambda) \}$ is finite.

Let $W$ be an open connected subset of $\Omega$ such that $W \cap F(\partial_{n-1} A) = \emptyset$. Then, there exists $k \in N$ such that the mapping $F: F^{-1}(W) \to W$ is a $k$-sheeted analytic covering, so that every $f \in A$ is holomorphic on $F^{-1}(W)$.

**Proof of Theorem 4.** By Corollary 3.1 there is a $k \in N$ such that $W = \bigcup_{i=1}^{k} W_i$, where $W_i = \{ \lambda \in W \mid g \text{ assumes i values on } F^{-1}(\lambda) \}$. Without loss of generality assume $W_i \neq \emptyset$. We shall show $g \circ F^{-1}$ is analytic in $W_k$. Let $\lambda \in W_k$ and $b_1, \ldots, b_k$ be the distinct values of $g$ on $F^{-1}(\lambda)$. Let $D_i \subset C (1 \leq i \leq k)$ be a disc centered at $b_i$. Assume $D_i \cap D_j = \emptyset$ for $i \neq j$.

**Assertion 1.** There exists a neighborhood $\mathcal{N}$ of $\lambda$ such that $g(F^{-1}(\mathcal{N})) \subset \bigcup_{i=1}^{k} D_i$ and $\overline{\mathcal{N}} \subset W$. This follows from continuity of $g$ and the topology of $M$.

**Assertion 2.** Let $e_i = F^{-1}(\mathcal{N}) \cap g^{-1}(\overline{D_i})$. Then, $F(e_i) = \overline{\mathcal{N}}$.

**Proof.** Denote $F|_{e_i} = F_i$. Apply Lemma 1 of [3] to $A F^{-1}(\overline{\mathcal{N}})$ and $e_i$ to obtain $\partial_{n-1} A_i e_i \subset F_i^{-1}(b \overline{\mathcal{N}})$. Recall that $\lambda \in C^n \setminus F(\partial_{n-1} A_i)$. If $J$ is a component with $\lambda \in J \subset C^n \setminus F(\partial_{n-1} A_i)$, by Lemma 2 of [3], $F(e_i) \cap J = J$. $J$ contains $\mathcal{N}$ and $F(e_i)$ is closed. So, $\mathcal{N} \subset \overline{\mathcal{N}} \subset F(e_i)$. We have $F(e_i) \subset \overline{\mathcal{N}}$ by the definition of $e_i$.

Since $g$ assumes at most $k$-values on every fiber, it is clear that $g$ is constant on the set $F^{-1}(\lambda) \cap e_i (\lambda \in \overline{\mathcal{N}}, 1 \leq i \leq k)$. In view of Lemma 3, $\lambda \mapsto g \circ F^{-1}(\lambda)$ is analytic on $N$. We have shown that $F^{-1}(W_k) \to W_k$ is a $k$-sheeted covering map, and also that $W_k$ is open.

Next, using Assertion 2 above and the ideas of Bishop and Basener [3, p. 103], we can show that $W \setminus W_k$ is a negligible set in $W$ and $F^{-1}(W_k)$ is dense in $F^{-1}(W)$.

Consequently, we conclude that $(F^{-1}(W), F, W)$ is a $k$-sheeted analytic cover in the sense of [6, p. 101]. This proves Theorem 4.

**Bibliography**


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