MULTI-DIMENSIONAL ANALYTIC STRUCTURE
AND SHILOV BOUNDARIES

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Abstract. We give a condition under which multi-dimensional analytic structure can be introduced into the maximal ideal space of a uniform algebra.

Introduction. Let $A$ be a uniform algebra on a compact Hausdorff space $X$ and $M$ the maximal ideal space of $A$. Various authors have showed that one-dimensional or multi-dimensional analytic structure can be introduced into $M$ provided that there exists a suitable subset $G$ of $C$, or $C^n$ respectively, with "the finite fibre property". Thus, the classic theorem on the subject by E. Bishop [5] states

**Theorem 1.** For $f \in A$, define $f^{-1}(\lambda) = \{x \in M: f(x) = \lambda\}$. Let $W$ be a component of $C \setminus f(X)$. Suppose that there exists a subset $G$ of $W$ such that:

1. $G$ has positive two-dimensional Lebesgue measure.
2. For each $\lambda$ in $G$, $\#f^{-1}(\lambda)$, the cardinality of $\{f^{-1}(\lambda)\}$, is finite.

Then, there is an integer $n$ such that for every $\lambda \in W$, $\#f^{-1}(\lambda) < n$. Furthermore, $f^{-1}(W)$ can be given the structure of a one-dimensional complex analytic space such that each $g$ in $A$ is holomorphic on this space.

B. Aupetit and J. Wermer [1] showed that the hypothesis on $G$ of "positive measure" can be replaced by "positive exterior logarithmic capacity", and no weaker condition will suffice.

Also generalizing Bishop’s result, R. Basener [3] and N. Sibony [9] independently formulated a condition for the existence of an $n$-dimensional analytic structure as follows: Let $A^n = \{(f_1, \ldots, f_n) | f_1, \ldots, f_n \in A\}$, so that each $F \in A^n$ maps $M$ to $C^n$. Let $V(F) = \{x \in M: F(x) = (0, \ldots, 0)\}$. The $n$th Shilov boundary $\partial_nA$ is defined by $\partial_nA = \text{closure} \{\bigcup_{F \in A^n} \partial_0A_{V(F)}\}$. $\partial_0A$ is the usual Shilov boundary.

**Theorem 2.** Fix $F \in A^n$. Let $W$ be a component of $F(M) \setminus F(\partial_{n-1}A)$. Suppose there exists $G \subseteq W$ such that:

1. $M_{2n}(G) > 0$ ($M_{2n}$ is the Lebesgue measure in $C^n$).
2. For each $\lambda \in G$, $\#F^{-1}(\lambda)$ is finite.
Then, there exists a positive integer \( k \) such that for all \( \lambda \in W \), \( \# F^{-1}(\lambda) \) is at most \( k \). Moreover, \( S = (F^{-1}(W), F, W) \) is a branched analytic cover; consequently \( F^{-1}(W) \) is an analytic space and for every \( f \in A \), \( f \) is holomorphic on \( F^{-1}(W) \).

When \( n = 1 \), this is Bishop’s theorem.

Recently B. Aupetit [2] improved Theorem 2 by replacing (1) with the condition that \( G \) is not pluri-polar. (\( G \) is not pluri-polar if there is no plurisubharmonic function \( \phi \) on \( C^n \) such that \( G \subseteq \{ \lambda \in C^n | \phi(\lambda) = -\infty \} \).) Aupetit’s proof of the above requires that \( G \) is contained in \( F(M) \setminus F(\partial_{n-1}A) \). By the definition of the \( n \)th Shilov boundary, we have \( \partial_0 A \subset \partial_0 A_1 \subset \cdots \subset M \). Since a component of \( F(M) \setminus F(\partial_{n-1}A) \) is open in \( C^n \) [3, Lemma 2], it is contained in the interior of \( F(M) \setminus F(\partial_0 A) \). In this paper we formulate a condition for an \( n \)-dimensional analytic structure assuming that \( G \) is a non-pluri-polar set contained in the interior of \( F(M) \setminus F(\partial_0 A) \). The hypothesis of “non-pluri-polar set” on \( G \) is then replaced by a more general “uniqueness set” (Let \( \Omega \) be a region in \( C^n \). \( G \subseteq \bar{\Omega} \) is a uniqueness set (for \( \Omega \)) if every plurisubharmonic function on \( \Omega \) that converges to \(-\infty \) at every point of \( G \) is identically equal to \(-\infty \) on \( \Omega \).)

Our main results are stated in Theorems 3 and 4. We make essential use of the plurisubharmonicity proof for a certain class of functions associated with a uniform algebra, which the author developed in [7] and extended in [8]. We give an example covered by Theorem 4 of this paper but not by the Basener-Sibony-Aupetit Theorem.

**Example.** Let \( \Delta^2 \) be the open unit bidisc in \( C^2 \) and \( A = A(\Delta^2) \). Then \( \partial_0 A = \{(z, w) \in \Delta^2: |z| = 1, |w| = 1\} \); \( \partial_1 A = \{(z, w) \in \Delta^2: |z| = 1 \text{ or } |w| = 1\} \); and \( \partial_2 A = \Delta^2 = M \). The set \( G = \{(z, w): |z| < 1, |w| = 1\} \) is a uniqueness set for \( \Delta^2 \) and its 4-dimensional Lebesgue measure is zero. If we take \( F \) to be the map \((z, w)\) then \( G \) is contained in \( F(M) \setminus F(\partial_0 A) \) but not in \( F(M) \setminus F(\partial_{n-1}A) \).

Theorem 4 can be readily extended to the case where \( \# \{ F^{-1}(\lambda) \} \) is assumed to be countable on \( G \) in view of Basener’s Theorem [4].

We introduce some definitions and notations. Fix \( F \in A^n \) and denote by \( \mathcal{M} \) the subset of the Cartesian product of \( n \)-copies of \( M \) consisting of the points \( m = (m_1, \ldots, m_n) \) such that \( F(m_1) = \cdots = F(m_n) \). Define the projection \( \pi \) on \( \mathcal{M} \) by \( \pi(m) = F(m_1) \). Let \( \mathcal{A} \) be the uniform algebra on \( \mathcal{M} \) generated by the functions of the form \( \theta \rightarrow g_1(\theta_1) \cdots g_n(\theta_n) \), \( g_i \in A \), \( i = 1, 2, \ldots, n \). The maximal ideal space of \( \mathcal{A} \) is \( \mathcal{M} \).

**Lemma 1.** Let \( \Omega \) be a component of the interior of \( F(M) \setminus F(\partial_0 A) \) in \( C^n \). For each \( \tau \in \mathcal{A} \), the function \( \phi \), defined by

\[
\phi(\lambda) = \log \left( \max \left\{ |\tau(\theta)| : \theta \in \pi^{-1}(\lambda) \right\} \right),
\]

is plurisubharmonic in \( \Omega \).

**Proof.** The upper semicontinuity of \( \phi \) is proved by a standard method (see [11, p. 139]). We must show that if \( L \) is a complex line contained in \( \Omega \) the restriction of \( \phi \) to \( L \) is subharmonic. Let \( D \) be a disc contained in \( L \). For some \( \alpha_{jk} \) and \( \gamma_k \) in \( C \),

\[
L = \bigcap_{k=1}^{n-1} \left( \lambda_1, \ldots, \lambda_n \right) \in \Omega \left\{ \sum_{j=1}^{n} \alpha_{jk} \lambda_j = \gamma_k \right\}.
\]
Put
\[ V = \left\{ m \in \mathcal{M} \mid \sum_{j=1}^{n} \alpha_{jk} f_j(m_1) = \gamma_k, \ k = 1, \ldots, n - 1 \right\}. \]

By \( \mathcal{A}_V \) we mean the restriction algebra, \( \mathcal{A}_V = \{ f \in C(V) \mid f \) is the uniform limit of functions in \( \mathcal{A} \}. \) The maximal ideal space of \( \mathcal{A}_V \) is \( V \). Choose a polynomial \( P \) with \( \phi \leq \text{Re} P \) on \( bD \). Then, for each \( \xi \) in \( bD \), \( \max_{\pi^{-1}(\xi)} |\tau| \cdot |e^{-P(\xi)}| \leq 1 \). We must show that \( \phi \leq \text{Re} P \) on \( \overline{D} \).

Fix \( z \in D \). There exists \( \theta \in \pi^{-1}(z) \) such that \( |\tau(\theta)| = \max_{\pi^{-1}(z)} |\tau| \). The function \( \theta \mapsto \tau(\theta) \cdot e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))} \), restricted to \( V \), is in \( \mathcal{A}_V \). We shall show in Lemma 2 that \( \pi^{-1}(D) \subseteq V \setminus \partial_0[\mathcal{A}_V] \). Assuming that Lemma 2 is true, by the local maximum modulus principle of \( \mathcal{A}_V \) applied to \( \pi^{-1}(D) \),
\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))}| \leq |\tau(\alpha)| \cdot |e^{-P(f_1(\alpha_1), \ldots, f_n(\alpha_1))}| \]
for some \( \alpha = (\alpha_1, \ldots, \alpha_n) \in b[\pi^{-1}(D)] \subseteq \pi^{-1}(bD) \). Hence
\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1), \ldots, f_n(\theta_n))}| \leq 1. \]
Thus, for an arbitrary \( z \in \overline{D} \),
\[ \phi(z) = \log \max \{ |\tau(\theta)| : \theta \in \pi^{-1}(z) \} \leq \text{Re} P(z). \]
This proves Lemma 1 assuming that \( \pi^{-1}(D) \subseteq V \setminus \partial_0[\mathcal{A}_V] \), which is to be proven.

Next we deduce an important consequence of Lemma 1.

**COROLLARY 1.1.** Fix \( g \in A, F \in A^n \). For each \( k \in \mathbb{N} \),
\[ \psi_{k,g}(\lambda) = \log \max \left\{ \prod_{1 \leq i < j \leq k} |g(\theta_i) - g(\theta_j)| : \theta_i, \theta_j \in F^{-1}(\lambda) \right\} \]
is plurisubharmonic on \( \Omega \).

**PROOF.** \( \Pi_{1 \leq i < j \leq k} (g(\theta_i) - g(\theta_j)) \in \mathcal{A} \). If \( \theta_1, \ldots, \theta_k \in F^{-1}(\lambda) \), then \( (\theta_1, \ldots, \theta_k) \in \pi^{-1}(\lambda) \). The plurisubharmonicity of \( \psi_{k,g} \) follows from Lemma 1.

**THEOREM 3.** Fix \( g \in A \) and \( F \in A^n \). Let \( \Omega \) be a component in the interior of \( F(M) \setminus F(0, A) \) in \( \mathbb{C}^n \). Suppose that there exists a subset \( G \) of \( \Omega \) with the following properties:

(i) \( G \) is not pluri-polar.

(ii) For every \( \lambda \in G \), \( \{ g \circ F^{-1}(\lambda) \} \) is finite.

Then, there exists \( k \in \mathbb{N} \) such that for each \( \lambda \in \Omega \), \( \{ g \circ F^{-1}(\lambda) \} \) is at most \( k \).

**PROOF.** The condition (ii) implies that \( G = \bigcup_{i \in \mathbb{N}} G_i \), where \( G_i = \{ \lambda \in G \mid g \) assumes \( i \) values on \( F^{-1}(\lambda) \} \). For some \( k \in \mathbb{N} \), \( G_k \) is non-pluri-polar. Since for each \( \lambda \in G_k \), \( g \) assume \( k \) values on \( F^{-1}(\lambda) \),
\[ \max_{\theta, \theta \in F^{-1}(\lambda)} \prod_{1 \leq i < j \leq k + 1} |g(\theta_i) - g(\theta_j)| = 0 \]
on \( G_k \). Hence, \( \psi_{k+1,g} \equiv -\infty \) on \( \Omega \). This implies that \( g \) assumes at most \( k \) values on \( F^{-1}(\lambda) \) for each \( \lambda \in \Omega \). Take \( k \) to be the largest integer such that \( \Omega_k \cap \Omega \neq \emptyset \). Thus, \( \Omega = \bigcup_{i=1}^{k} \Omega_i \).
We shall now prove the hypothesis used in Lemma 1.

**Lemma 2. Use notations as in Lemma 1.**

\[
\pi^{-1}(D) \subseteq V \setminus \partial_0[\mathcal{V}_V].
\]

**Proof.** Let \( s \in \pi^{-1}(D) \). Then \( \pi(s) = F(s_1) = (f_1(s_1), \ldots, f_n(s_1)) \in D \subseteq L \). Therefore, \( s \in V \). We need to show \( s \notin \partial_0[\mathcal{V}_V] \). Put \( a = \pi(s) \). Let \( U \) be an open disc contained in \( L \) and centered at \( a \). Let \( bU \) be the boundary of \( U \) in the topology of \( L \).

For each function \( H \in \mathcal{V}_V \) we shall construct a bounded analytic function \( \gamma \) on \( U \) satisfying \( H(s) = \gamma(a) \) and \( |\gamma(a)| \leq \max_{\pi^{-1}(bU)} |H| \). We may assume without loss of generality that \( H \) takes the form,

\[
H(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} h_{ij}(\theta_j); \quad (\theta_1, \ldots, \theta_n) \in V, l \in N, h_{ij} \in \mathcal{A}.
\]

Let \( U_j \) be the projection of \( U \) on the \( j \)-th coordinate axis. \( F^{-1}(\overline{U}) = \bigcap_{j=1}^{l} \{ f_j^{-1}(\overline{U}_j) \} \) and it is \( \mathcal{A} \)-convex. Moreover, \( F^{-1}(\overline{U}) \cap \partial_0 \mathcal{A} = \emptyset \). Let \( f_j \) be the restriction of \( f_i \) to \( F^{-1}(\overline{U}) \). By the local maximum modulus principle applied to \( A \) with respect to \( F^{-1}(\overline{U}), \partial_0 A_{F^{-1}(\overline{U})} \subseteq \overline{f_j^{-1}(bU_j)} \). Let \( \mu_j \) be a representing measure for \( s_j \) concentrated on \( \partial_0 A_{F^{-1}(\overline{U})} \). For each \( h_{ij} \) in (1), define

\[
\eta_{ij}(z) = \int_{f_j^{-1}(bU_j)} \frac{f_j - a_j}{f_j - z} h_{ij} \, d\mu_j.
\]

**Assertion.** \( \eta_{ij} \) has the following properties:

(i) \( \eta_{ij} \) is bounded and analytic on \( U_j \).

(ii) If \( \xi_j \in bU_j \) and a nontangential limit \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z) \) exists, then

\[
|\eta_{ij}(\xi_j)| \leq \max_{f_j^{-1}(\xi_j)} |h_{ij}|.
\]

(iii) \( \eta_{ij}(a_j) = h_{ij}(s_j) \).

The proof for the assertion is the same as that used by Seničkin in [10, Lemma 7]. Using the functions, \( \eta_{ij} \) defined above, form a bounded analytic function \( \gamma \) on \( U \) as

\[
\gamma(z_1, \ldots, z_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} \eta_{ij}(z_j).
\]

Choose \( \xi_j \in bU_j \) so that all the nontangential limits \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z), z \in U_j, 1 \leq i \leq l \), exist, and \( (\xi_1, \ldots, \xi_n) \in bU \). Put

\[
\gamma(\xi_1, \ldots, \xi_n) = \lim_{(z_1, \ldots, z_n) \to (\xi_1, \ldots, \xi_n)} \gamma(z_1, \ldots, z_n).
\]

Using essentially the same proof as in [8, Lemma 2] we obtain

\[
|\gamma(\xi_1, \ldots, \xi_n)| \leq \max_{\pi^{-1}(bU)} |H|.
\]

This is true for almost all points in \( bU \) and \( \gamma \) is analytic. So,

\[
|\gamma(a_1, \ldots, a_n)| \leq \max_{\pi^{-1}(bU)} |H|.
\]
Hence
\[ |H(s)| \leq \max_{x \in \mathbb{R}} |H|. \]

Since \( s \) and \( H \) were chosen arbitrarily and \( \partial_0[\mathfrak{H}_s] \) is the closure of the generalized peak points of \( \mathfrak{H}_s \), this yields the desired conclusion
\[ \partial_0[\mathfrak{H}_s] \cap \pi^{-1}(D) = \emptyset. \]

Theorem 3 can be extended, using the same proof, as follows.

**Definition.** Let \( \Omega \) be a region on \( \mathbb{C}^n \). We say \( G \subseteq \Omega \) is a set of uniqueness for \( \Omega \) if every plurisubharmonic function defined on \( \Omega \) that converges to \( -\infty \) at every point of \( G \) is identically equal to \( -\infty \) on \( \Omega \).

**Corollary 3.1.** Let \( \Omega \) be a component of the interior of \( F(M) \setminus F(\partial_0 A) \) and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \) of \( \overline{\Omega} \) satisfying
(i) \( G \) is a set of uniqueness for \( \Omega \).
(ii) For every \( \lambda \in G \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is finite.

Then, there exists \( k \in \mathbb{N} \) such that for each \( \lambda \) in \( \Omega \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is at most \( k \).

The following lemma is a special case of Theorem 4.

**Lemma 3.** Let \( A, M, \) and \( F \) be as before. Suppose that \( W \) is a component of \( F(M) \setminus F(\partial_{n-1} A) \), and suppose \( g \in A \) is constant on \( F_M'(A) \) for every \( A \in W \). Then, \( g \circ F^{-1} \) is analytic on \( W \).

**Proof.** We show \( g \circ F^{-1} \) is analytic in each variable. Let \( a \in W \) and \( \Delta''(a, r) \) be an open polydisc about \( a \); \( \Delta''(a, r) = \prod_{i=1}^n \Delta_i, \Delta_i = \Delta(a_i, r_i) \). Put \( \Delta_1 = \{ (a_1, \ldots, z_i, \ldots, a_n) | z_i \in \Delta_i \} \). Note that \( F^{-1}(\Delta_1) = f^{-1}(\Delta_i) \cap \Gamma \), where \( \Gamma \) is the set of zeros of \( n-1 \) functions in \( A \). For simplicity of notation denote by \( f \) the restriction of \( f_i \) to \( \Gamma \). Thus \( F^{-1}(\Delta_i) = f^{-1}(\Delta_i) \), and we shall show that \( g \circ f^{-1} \) is analytic on \( \Delta_i \).

Consider the algebra \( \mathcal{A}_\Gamma \). By the definition of \( \Delta_{n-1} A \), we have \( \partial_0 A \subseteq \Delta_{n-1} A \), and \( \Delta_i \cap f(\partial_0 A_{\Gamma}) = \emptyset \) since \( \Delta(a, r) \cap F(\partial_{n-1} A) = \emptyset \). Note that \( \partial_0 A_{f^{-1}(\Delta_i)} \subseteq f^{-1}(b\Delta_i) \), by the local maximum modulus principle of \( A_{\Gamma} \) applied to \( f^{-1}(\Delta_i) \).

Let \( \mu_i \) be the representing measure for some \( m_i \in f^{-1}(a_i) \) supported on \( \partial_0 A_{f^{-1}(\Delta_i)} \), and \( \nu_i \) the projection of \( \mu_i \) on \( b\Delta_i \), which is the normalized Lebesgue measure on \( b\Delta_i \).

\[
\int_{b\Delta_i} g \circ f^{-1} d\nu_i = \int_{f^{-1}(b\Delta_i)} gd\mu_i = g(m_i) = g \circ f^{-1}(a_i).
\]

Thus, \( g \circ f^{-1} \) is a complex harmonic function.

\[
\int_{b\Delta_i} (z - a_i)^n \cdot g \circ f^{-1} d\nu_i = \int_{f^{-1}(b\Delta_i)} (f - a_i)^n \cdot gd\mu_i = 0 \quad (n \in \mathbb{N}).
\]

This shows that \( g \circ f^{-1} \) is holomorphic on \( \Delta_i \), and hence, \( g \circ F^{-1} \) on \( \Delta_i \).

**Theorem 4.** Fix \( F \in A^n \) and \( g \in A \). Let \( \Omega \) be a component contained in the interior of \( F(M) \setminus F(\partial_0 A) \), and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \subseteq \overline{\Omega} \) such that:

(i) \( G \) is a set of uniqueness for \( \Omega \).
(ii) For every $\lambda \in G$, $\# \{g \circ F^{-1}(\lambda)\}$ is finite.

Let $W$ be an open connected subset of $\Omega$ such that $W \cap F(\partial_{n-1}A) = \emptyset$. Then, there exists $k \in \mathbb{N}$ such that the mapping $F: F^{-1}(W) \to W$ is a $k$-sheeted analytic covering, so that every $f \in A$ is holomorphic on $F^{-1}(W)$.

**Proof of Theorem 4.** By Corollary 3.1 there is a $k \in \mathbb{N}$ such that $W = \bigcup_{i=1}^{k} W_i$, where $W_i = \{\lambda \in W | g$ assumes $i$ values on $F^{-1}(\lambda)\}$. Without loss of generality assume $W_k \neq \emptyset$. We shall show $g \circ F^{-1}$ is analytic in $W_k$. Let $\lambda \in W_k$ and $b_1, \ldots, b_k$ be the distinct values of $g$ on $F^{-1}(\lambda)$. Let $D_i \subset C (1 \leq i \leq k)$ be a disc centered at $b_i$. Assume $D_i \cap D_j = \emptyset$ for $i \neq j$.

**Assertion 1.** There exists a neighborhood $N$ of $\lambda$ such that $g(F^{-1}(\bar{N})) \subset \bigcup_{i=1}^{k} D_i$ and $\bar{N} \subseteq W$. This follows from continuity of $g$ and the topology of $M$.

**Assertion 2.** Let $\varepsilon_i = F^{-1}(\bar{N}) \cap g^{-1}(\bar{D}_i)$. Then, $F(\varepsilon_i) = \bar{N}$.

**Proof.** Denote $F|_{\varepsilon_i} = F_i$. Apply Lemma 1 of [3] to $A_{F^{-1}(\bar{N})}$ and $\varepsilon_i$ to obtain $\partial_{n-1} A_{\varepsilon_i} \subseteq F_i^{-1}(b\bar{N})$. Recall that $\lambda \in C^n \setminus F(\partial_{n-1} A_{\varepsilon_i})$. If $J$ is a component with $\lambda \in J \subset C^n \setminus F(\partial_{n-1} A_{\varepsilon_i})$, by Lemma 2 of [3], $F(\varepsilon_i) \cap J = J$. $J$ contains $N$ and $F(\varepsilon_i)$ is closed. So, $N \subseteq \bar{N} \subseteq F(\varepsilon_i)$. We have $F(\varepsilon_i) \subseteq \bar{N}$ by the definition of $\varepsilon_i$.

Since $g$ assumes at most $k$-values on every fiber, it is clear that $g$ is constant on the set $F^{-1}(\lambda) \cap \varepsilon_i (\lambda \in \bar{N}, 1 \leq i \leq k)$. In view of Lemma 3, $\lambda \mapsto g \circ F^{-1}(\lambda)$ is analytic on $N$. We have shown that $F^{-1}(W_k) \to W_k$ is a $k$-sheeted covering map, and also that $W_k$ is open.

Next, using Assertion 2 above and the ideas of Bishop and Basener [3, p. 103], we can show that $W \setminus W_k$ is a negligible set in $W$ and $F^{-1}(W_k)$ is dense in $F^{-1}(W)$.

Consequently, we conclude that $(F^{-1}(W), F, W)$ is a $k$-sheeted analytic cover in the sense of [6, p. 101]. This proves Theorem 4.

**Bibliography**


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