MULTI-DIMENSIONAL ANALYTIC STRUCTURE
AND SHILOV BOUNDARIES

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Abstract. We give a condition under which multi-dimensional analytic structure
can be introduced into the maximal ideal space of a uniform algebra.

Introduction. Let $A$ be a uniform algebra on a compact Hausdorff space $X$ and $M$
the maximal ideal space of $A$. Various authors have showed that one-dimensional or
multi-dimensional analytic structure can be introduced into $M$ provided that there
exists a suitable subset $G$ of $C$, or $C^n$ respectively, with "the finite fibre property".
Thus, the classic theorem on the subject by E. Bishop [5] states

Theorem 1. For $f \in A$, define $f^{-1}(\lambda) = \{x \in M: f(x) = \lambda\}$. Let $W$ be a compo-
nent of $C \setminus f(X)$. Suppose that there exists a subset $G$ of $W$ such that:

1. $G$ has positive two-dimensional Lebesgue measure.
2. For each $\lambda$ in $G$, $\#f^{-1}(\lambda)$, the cardinality of $\{f^{-1}(\lambda)\}$, is finite.

Then, there is an integer $n$ such that for every $\lambda \in W$, $\#f^{-1}(\lambda) \leq n$. Furthermore,
$f^{-1}(W)$ can be given the structure of a one-dimensional complex analytic space such
that each $g$ in $A$ is holomorphic on this space.

B. Aupetit and J. Wermer [1] showed that the hypothesis on $G$ of "positive
measure" can be replaced by "positive exterior logarithmic capacity", and no weaker
condition will suffice.

Also generalizing Bishop’s result, R. Basener [3] and N. Sibony [9] independently
formulated a condition for the existence of an $n$-dimensional analytic structure as
follows: Let $A^n = \{(f_1, \ldots, f_n)| f_1, \ldots, f_n \in A\}$, so that each $F \in A^n$ maps $M$ to $C^n$.
Let $V(F) = \{x \in M: F(x) = (0, \ldots, 0)\}$. The $n$th Shilov boundary $\partial_n A$ is defined
by $\partial_n A = \text{closure}[\bigcup_{F \in A^n} \partial_0 A_{V(F)}]$, $\partial_0 A$ is the usual Shilov boundary.

Theorem 2. Fix $F \in A^n$. Let $W$ be a component of $F(M) \setminus F(\partial_{n-1} A)$. Suppose
there exists $G \subseteq W$ such that:

1. $M_{2n}(G) > 0$ ($M_{2n}$ is the Lebesgue measure in $C^n$).
2. For each $\lambda \in G$, $\#F^{-1}(\lambda)$ is finite.
Then, there exists a positive integer $k$ such that for all $\lambda \in W$, $\#F^{-1}(\lambda)$ is at most $k$. Moreover, $S = (F^{-1}(W), F, W)$ is a branched analytic cover; consequently $F^{-1}(W)$ is an analytic space and for every $f \in A, f$ is holomorphic on $F^{-1}(W)$.

When $n = 1$, this is Bishop's theorem.

Recently B. Aupetit [2] improved Theorem 2 by replacing (1) with the condition that $G$ is not pluri-polar. ($G$ is not pluri-polar if there is no plurisubharmonic function $\phi$ on $C^n$ such that $G \subseteq \{ \lambda \in C^n | \phi(\lambda) = -\infty \}$.)

Aupetit's proof of the above requires that $G$ is contained in $F(M) \setminus F(\partial_{n-1}A)$. By the definition of the $n$th Shilov boundary, we have $\partial_{n}A \subseteq \partial_{n-1}A \subseteq \cdots \subseteq M$. Since a component of $F(M) \setminus F(\partial_{n-1}A)$ is open in $C^n$ [3, Lemma 2], it is contained in the interior of $F(M) \setminus F(\partial_0A)$. In this paper we formulate a condition for an $n$-dimensional analytic structure assuming that $G$ is a non-pluri-polar set contained in the interior of $F(M) \setminus F(\partial_{n}A)$. The hypothesis of "non-pluri-polar set" on $G$ is then replaced by a more general "uniqueness set". (Let $\Omega$ be a region in $C^n$. $G \subseteq \Omega$ is a uniqueness set (for $\Omega$) if every plurisubharmonic function on $\Omega$ that converges to $-\infty$ at every point of $G$ is identically equal to $-\infty$ on $\Omega$.)

Our main results are stated in Theorems 3 and 4. We make essential use of the plurisubharmonicity proof for a certain class of functions associated with a uniform algebra, which the author developed in [7] and extended in [8]. We give an example covered by Theorem 4 of this paper but not by the Basener-Sibony-Aupetit Theorem.

**Example.** Let $\Delta^2$ be the open unit bidisc in $C^2$ and $A = A(\Delta^2)$. Then $\partial_0A = \{ (z, w) \in \Delta^2 : |z| = 1, |w| = 1 \}$; $\partial_1A = \{ (z, w) \in \Delta^2 : |z| = 1 \text{ or } |w| = 1 \}$; and $\partial_2A = \Delta^2 = M$. The set $G = \{ (z, w) : |z| < 1, |w| = 1 \}$ is a uniqueness set for $\Delta^2$ and its 4-dimensional Lebesgue measure is zero. If we take $F$ to be the map $(z, w)$ then $G$ is contained in $F(M) \setminus F(\partial_0A)$ but not in $F(M) \setminus F(\partial_{n-1}A)$.

Theorem 4 can be readily extended to the case where $\# \{ F^{-1}(\lambda) \}$ is assumed to be countable on $G$ in view of Basener’s Theorem [4].

We introduce some definitions and notations. Fix $F \in A^n$ and denote by $\mathcal{M}$ the subset of the Cartesian product of $n$-copies of $M$ consisting of the points $m = (m_1, \ldots, m_n)$ such that $F(m_1) = \cdots = F(m_n)$. Define the projection $\pi$ on $\mathcal{M}$ by $\pi(m) = F(m_1)$. Let $\mathcal{A}$ be the uniform algebra on $\mathcal{M}$ generated by the functions of the form $\theta \rightarrow g_1(\theta_1) \cdots g_n(\theta_n), g_i \in A, i = 1, 2, \ldots, n$. The maximal ideal space of $\mathcal{A}$ is $\Omega$.

**Lemma 1.** Let $\Omega$ be a component of the interior of $F(M) \setminus F(\partial_0A)$ in $C^n$. For each $\tau \in \mathcal{A}$, the function $\phi$, defined by

$$\phi(\lambda) = \log \left\{ \max_\tau(\theta) : \theta \in \pi^{-1}(\lambda) \right\},$$

is plurisubharmonic in $\Omega$.

**Proof.** The upper semicontinuity of $\phi$ is proved by a standard method (see [11, p. 139]). We must show that if $L$ is a complex line contained in $\Omega$ the restriction of $\phi$ to $L$ is subharmonic. Let $D$ be a disc contained in $L$. For some $\alpha_{jk}$ and $\gamma_k$ in $C$,

$$L = \bigcap_{k=1}^{n-1} \left\{ (\lambda_1, \ldots, \lambda_n) \in \Omega : \sum_{j=1}^{n} \alpha_{jk} \lambda_j = \gamma_k \right\}.$$
Put

\[ V = \left\{ m \in \mathcal{M} \mid \sum_{j=1}^{n} \alpha_{jk} f_j(m_1) = \gamma_k, \ k = 1, \ldots, n - 1 \right\}. \]

By $\mathcal{A}_V$ we mean the restriction algebra, $\mathcal{A}_V = \{ f \in C(V) \mid f$ is the uniform limit of functions in $\mathcal{A} \}$

The maximal ideal space of $\mathcal{A}_V$ is $V$. Choose a polynomial $P$ with $\phi \leq \text{Re} \ P \text{ on } bD$. Then, for each $\xi$ in $bD$, max$_{\pi^{-1}(\xi)}|\tau| \cdot |e^{-P(\xi)}| \leq 1$. We must show that $\phi \leq \text{Re} \ P$ on $\overline{D}$.

Fix $z \in D$. There exists $\theta \in \pi^{-1}(z)$ such that $|\tau(\theta)| = \max_{\pi^{-1}(z)}|\tau|$. The function $\theta \rightarrow \tau(\theta) \cdot e^{-P(f_1(\theta_1) + \ldots + f_n(\theta_n))}$, restricted to $V$, is in $\mathcal{A}_V$. We shall show in Lemma 2 that $\pi^{-1}(D) \subseteq V \ \partial_0[\mathcal{A}_V]$. Assuming that Lemma 2 is true, by the local maximum modulus principle of $\mathcal{A}_V$, applied to $\pi^{-1}(D)$,

\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1) + \ldots + f_n(\theta_n))}| \leq |\tau(\alpha)| \cdot |e^{-P(f_1(\alpha_1) + \ldots + f_n(\alpha_n))}| \]

for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in b[\pi^{-1}(D)] \subseteq \pi^{-1}(bD)$. Hence

\[ |\tau(\theta)| \cdot |e^{-P(f_1(\theta_1) + \ldots + f_n(\theta_n))}| \leq 1. \]

Thus, for an arbitrary $z \in \overline{D}$,

\[ \phi(z) = \log \max \{ |\tau(\theta)| : \theta \in \pi^{-1}(z) \} \leq \text{Re} P(z). \]

This proves Lemma 1 assuming that $\pi^{-1}(D) \subseteq V \ \partial_0[\mathcal{A}_V]$, which is to be proven.

Next we deduce an important consequence of Lemma 1.

**Corollary 1.1.** Fix $g \in A$, $F \in A^n$. For each $k \in N$,

\[ \psi_{k,g}(\lambda) = \log \max \left\{ \prod_{1 \leq i < j \leq k} |g(\theta_i) - g(\theta_j)| : \theta_i, \theta_j \in F^{-1}(\lambda) \right\} \]

is plurisubharmonic on $\Omega$.

**Proof.** $\prod_{1 \leq i < j \leq k} (g(\theta_i) - g(\theta_j)) \in \mathcal{A}$. If $\theta_1, \ldots, \theta_k \in F^{-1}(\lambda)$, then $(\theta_1, \ldots, \theta_k) \in \pi^{-1}(\lambda)$. The plurisubharmonicity of $\psi_{k,g}$ follows from Lemma 1.

**Theorem 3.** Fix $g \in A$ and $F \in A^n$. Let $\Omega$ be a component in the interior of $F(M) \setminus F(0, A)$ in $C^n$. Suppose that there exists a subset $G$ of $\Omega$ with the following properties:

(i) $G$ is not pluri-polar.
(ii) For every $\lambda \in G$, $\# \{ g \circ F^{-1}(\lambda) \}$ is finite.

Then, there exists $k \in N$ such that for each $\lambda \in \Omega$, $\# \{ g \circ F^{-1}(\lambda) \}$ is at most $k$.

**Proof.** The condition (ii) implies that $G = \bigcup_{i \in N} G_i$, where $G_i = \{ \lambda \in G \mid g$ assumes $i$ values on $F^{-1}(\lambda) \}$. For some $k \in N$, $G_k$ is non-pluri-polar. Since for each $\lambda \in G_k$, $g$ assume $k$ values on $F^{-1}(\lambda)$,

\[ \max_{\theta_i, \theta_j \in F^{-1}(\lambda)} \prod_{1 \leq i < j \leq k + 1} |g(\theta_i) - g(\theta_j)| = 0 \]

on $G_k$. Hence, $\psi_{k+1,g} \equiv -\infty$ on $\Omega$. This implies that $g$ assumes at most $k$ values on $F^{-1}(\lambda)$ for each $\lambda \in \Omega$. Take $k$ to be the largest integer such that $\Omega_k \cap \Omega \neq \emptyset$. Thus, $\Omega = \bigcup_{i=1}^{k} \Omega_i$. 

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We shall now prove the hypothesis used in Lemma 1.

**Lemma 2.** Use notations as in Lemma 1.

\[ \pi^{-1}(D) \subseteq V \setminus \partial_0[\mathbb{H}_V]. \]

**Proof.** Let \( s \in \pi^{-1}(D) \). Then \( \pi(s) = F(s_1, \ldots, s_n) \in D \subseteq L \). Therefore, \( s \in V \). We need to show \( s \not\in \partial_0[\mathbb{H}_V] \). Put \( a = \pi(s) \). Let \( U \) be an open disc contained in \( L \) and centered at \( a \). Let \( bU \) be the boundary of \( U \) in the topology of \( L \). For each function \( H \) in \( \mathbb{H}_V \), we shall construct a bounded analytic function \( \gamma \) on \( U \) satisfying \( H(s) = \gamma(a) \) and \( |\gamma(a)| \leq \max_{\pi^{-1}(bU)} |H| \). We may assume without loss of generality that \( H \) takes the form,

\[
(1) \quad H(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} h_{ij}(\theta_j); \quad (\theta_1, \ldots, \theta_n) \in V, l \in N, h_{ij} \in A.
\]

Let \( U_j \) be the projection of \( U \) on the \( j \)th coordinate axis. \( F^{-1}([0,1]) = \bigcap_{j=1}^{l} \{ f^{-1}_j([0,1]) \} \) and it is \( A \)-convex. Moreover, \( F^{-1}([0,1]) \cap \partial_0 A = \emptyset \). Let \( f_j \) be the restriction of \( f_i \) to \( F^{-1}(U_j) \). By the local maximum modulus principle applied to \( A \) with respect to \( F^{-1}(U_j) \), \( \partial_0 A F^{-1}(U_j) \subseteq f_j^{-1}(bU_j) \). Let \( \mu_j \) be a representing measure for \( s_j \) concentrated on \( \partial_0 A F^{-1}(U_j) \). For each \( h_{ij} \) in (1), define

\[ \eta_{ij}(z) = \int_{f_j^{-1}(bU_j)} \frac{f_j - a_j}{f_j - z} h_{ij} d\mu_j. \]

**Assertion.** \( \eta_{ij} \) has the following properties:

(i) \( \eta_{ij} \) is bounded and analytic on \( U_j \).

(ii) If \( \xi_j \in bU_j \) and a nontangential limit \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z) \) exists, then

\[ |\eta_{ij}(\xi_j)| \leq \max_{f_j^{-1}(\xi_j)} |h_{ij}|. \]

(iii) \( \eta_{ij}(a_j) = h_{ij}(s_j) \).

The proof for the assertion is the same as that used by Seničkin in [10, Lemma 7]. Using the functions, \( \eta_{ij} \) defined above, form a bounded analytic function \( \gamma \) on \( U \) as

\[ \gamma(z_1, \ldots, z_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} \eta_{ij}(z_j). \]

Choose \( \xi_j \in bU \) so that all the nontangential limits \( \eta_{ij}(\xi_j) = \lim_{z \to \xi_j} \eta_{ij}(z), z \in U_j \), \( 1 \leq i \leq l \), exist, and \( (\xi_1, \ldots, \xi_n) \in bU \). Put

\[ \gamma(\xi_1, \ldots, \xi_n) = \lim_{(z_1, \ldots, z_n) \to (\xi_1, \ldots, \xi_n)} \gamma(z_1, \ldots, z_n). \]

Using essentially the same proof as in [8, Lemma 2] we obtain

\[ |\gamma(\xi_1, \ldots, \xi_n)| \leq \max_{\pi^{-1}(bU)} |H|. \]

This is true for almost all points in \( bU \) and \( \gamma \) is analytic. So,

\[ |\gamma(a_1, \ldots, a_n)| \leq \max_{\pi^{-1}(bU)} |H|. \]
Hence
\[ |H(s)| \leq \max_{\mathcal{H}} |H|. \]

Since \( s \) and \( H \) were chosen arbitrarily and \( \partial_0[\mathcal{H}] \) is the closure of the generalized peak points of \( \mathcal{H} \), this yields the desired conclusion
\[ \partial_0[\mathcal{H}] \cap \pi^{-1}(D) = \emptyset. \]

Theorem 3 can be extended, using the same proof, as follows.

**Definition.** Let \( \Omega \) be a region on \( \mathbb{C}^n \). We say \( G \subseteq \Omega \) is a set of uniqueness for \( \Omega \) if every plurisubharmonic function defined on \( \Omega \) that converges to \(-\infty\) at every point of \( G \) is identically equal to \(-\infty\) on \( \Omega \).

**Corollary 3.1.** Let \( \Omega \) be a component of the interior of \( F(M) \setminus F(\partial_0 A) \) and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \subseteq \overline{\Omega} \) satisfying

(i) \( G \) is a set of uniqueness for \( \Omega \).
(ii) For every \( \lambda \in G \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is finite.

Then, there exists \( k \in N \) such that for each \( \lambda \in \Omega \), \( \# \{ g \circ F^{-1}(\lambda) \} \) is at most \( k \).

The following lemma is a special case of Theorem 4.

**Lemma 3.** Let \( A, M, \) and \( F \) be as before. Suppose that \( W \) is a component of \( F(M) \setminus F(\partial_{n-1} A) \), and suppose \( g \in A \) is constant on \( F^{-1}(A) \) for every \( A \in W \). Then, \( g \circ F^{-1} \) is analytic on \( W \).

**Proof.** We show \( g \circ F^{-1} \) is analytic in each variable. Let \( a \in W \) and \( \Delta(a, r) \) be an open polydisc about \( a \); \( \Delta(a, r) = \prod_{i=1}^{n} \Delta_i \), \( \Delta_i = \Delta(a_i, r_i) \). Put \( g_i = (a_1, \ldots, z_i, \ldots, a_n) \). Note that \( F^{-1}(\Delta_i) = f_i^{-1}(\Delta_i) \cap \Gamma \), where \( \Gamma \) is the set of zeros of \( n-1 \) functions in \( A \). For simplicity of notation denote by \( f \) the restriction of \( f \) to \( \Gamma \). Thus \( F^{-1}(\Delta_i) = f^{-1}(\Delta_i) \), and we shall show that \( g \circ f^{-1} \) is analytic on \( \Delta_i \).

Consider the algebra \( A_\Gamma \). By the definition of \( \partial_{n-1} A \), we have \( \partial_0 A_\Gamma \subseteq \partial_{n-1} A \), and \( \Delta_i \cap \partial_0 A_\Gamma = \emptyset \) since \( \Delta(a, r) \cap F(\partial_{n-1} A) = \emptyset \). Note that \( \partial_0 A_\Gamma \cap \Delta_i \subseteq f^{-1}(b\Delta_i) \), by the local maximum modulus principle of \( A_\Gamma \) applied to \( f^{-1}(\Delta_i) \). Let \( \mu_i \) be the representing measure for some \( m_i \in f^{-1}(a_i) \) supported on \( \partial_0 A_\Gamma(\Delta_i) \), and \( \nu_i \) the projection of \( \mu_i \) on \( b\Delta_i \), which is the normalized Lebesgue measure on \( b\Delta_i \).

\[
\int_{b\Delta_i} g \circ f^{-1} dv_i = \int_{f^{-1}(b\Delta_i)} g d\mu_i = g(m_i) = g \circ f^{-1}(a_i).
\]

Thus, \( g \circ f^{-1} \) is a complex harmonic function.

\[
\int_{b\Delta_i} (z-a_i)^n \cdot g \circ f^{-1} dv_i = \int_{f^{-1}(b\Delta_i)} (f-a_i)^n \cdot g d\mu_i = 0 \quad (n \in N).
\]

This shows that \( g \circ f^{-1} \) is holomorphic on \( \Delta_i \), and hence, \( g \circ F^{-1} \) on \( \Delta' \).

**Theorem 4.** Fix \( F \in A^n \) and \( g \in A \). Let \( \Omega \) be a component contained in the interior of \( F(M) \setminus F(\partial_0 A) \), and \( \overline{\Omega} \) its closure in \( F(M) \setminus F(\partial_0 A) \). Suppose there exists a subset \( G \subseteq \overline{\Omega} \) such that:

(i) \( G \) is a set of uniqueness for \( \Omega \).
(ii) For every $\lambda \in G$, $\# \{ g \circ F^{-1}(\lambda) \}$ is finite.

Let $W$ be an open connected subset of $\Omega$ such that $W \cap F(\partial_{n-1}A) = \emptyset$. Then, there exists $k \in N$ such that the mapping $F: F^{-1}(W) \rightarrow W$ is a $k$-sheeted analytic covering, so that every $f \in A$ is holomorphic on $F^{-1}(W)$.

**Proof of Theorem 4.** By Corollary 3.1 there is a $k \in N$ such that $W = \bigcup_{i=1}^{k} W_i$, where $W_i = \{ \lambda \in W | g$ assumes $i$ values on $F^{-1}(\lambda) \}$. Without loss of generality assume $W_k \neq \emptyset$. We shall show $g \circ F^{-1}$ is analytic in $W_k$. Let $\lambda \in W_k$ and $b_1, \ldots, b_k$ be the distinct values of $g$ on $F^{-1}(\lambda)$. Let $D_i \subset C (1 \leq i \leq k)$ be a disc centered at $b_i$. Assume $\bar{D}_i \cap \bar{D}_j = \emptyset$ for $i \neq j$.

**Assertion 1.** There exists a neighborhood $N$ of $\lambda$ such that $g(F^{-1}(\bar{N})) \subset \bigcup_{i=1}^{k} D_i$ and $\bar{N} \subset W$. This follows from continuity of $g$ and the topology of $M$.

**Assertion 2.** Let $e_i = F^{-1}(\bar{N}) \cap g^{-1}(\bar{D}_i)$. Then, $F(e_i) = \bar{N}$.

**Proof.** Denote $F|_{e_i} = F_i$. Apply Lemma 1 of [3] to $A F^{-1}(\bar{N})$ and $e_i$ to obtain $\partial_{n-1} A e_i \subset F_i^{-1}(bN)$. Recall that $\lambda \in C^n \setminus F(\partial_{n-1} A e_i)$. If $J$ is a component with $\lambda \in J \subset C^n \setminus F(\partial_{n-1} A e_i)$, by Lemma 2 of [3], $F(e_i) \cap J = J$. $J$ contains $N$ and $F(e_i)$ is closed. So, $N \subset \bar{N} \subset F(e_i)$. We have $F(e_i) \subset \bar{N}$ by the definition of $e_i$.

Since $g$ assumes at most $k$-values on every fiber, it is clear that $g$ is constant on the set $F^{-1}(\lambda) \cap e_i (\lambda \in \bar{N}, 1 \leq i \leq k)$. In view of Lemma 3, $\lambda \mapsto g \circ F^{-1}(\lambda)$ is analytic on $N$. We have shown that $F^{-1}(W_k) \rightarrow W_k$ is a $k$-sheeted covering map, and also that $W_k$ is open.

Next, using Assertion 2 above and the ideas of Bishop and Basener [3, p. 103], we can show that $W \setminus W_k$ is a negligible set in $W$ and $F^{-1}(W_k)$ is dense in $F^{-1}(W)$.

Consequently, we conclude that $(F^{-1}(W), F, W)$ is a $k$-sheeted analytic cover in the sense of [6, p. 101]. This proves Theorem 4.

**Bibliography**


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