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ω-CONNECTED CONTINUA AND JONES' K FUNCTION

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ABSTRACT. A continuum X is *ω-connected* if for every pair of points x, y of X, there exists an irreducible subcontinuum of X from x to y that is decomposable. If $A \subset X$ then $K(A)$ is the intersection of all subcontinua of X that contain A in their interiors. The main theorem shows that if X is an ω-connected continuum and H is a connected nowhere dense subset of X, then $K(H)$ has a void interior. Several corollaries are established for continua with certain separation properties and a final theorem shows the equivalence of ω-connectedness and δ-connectedness for plane continua.

Let X be a compact connected metric space (continuum). If for every pair of distinct points x, y of X there exists a subcontinuum I of X irreducible between x and y such that; (1) I is decomposable, then X is *ω-connected* [**3**]; (2) I contains no indecomposable subcontinuum with nonvoid interior relative to I, then X is *λ-connected* [**7**]; (3) I is hereditarily decomposable, then I is *δ-connected* [**7**]; (4) I is an arc, then I is *α-connected* (arcwise connected). The first three properties are generalizations of arcwise connectedness and it is clear that α-connectedness implies δ-connectedness which implies λ-connectedness which implies ω-connectedness. Also if $A \subset X$ let $K(A)$ be the intersection of all subcontinua of X that contain A in their interiors relative to X. This concept was introduced by F. B. Jones in [**9**, Theorem 2]. There the K function is restricted to points (rather than subsets).

In [**10**] H. E. Schlais proves the following:

THEOREM. *If X is a hereditarily decomposable continuum, then for every point x of X, the interior of $K(x)$ relative to X is void.*

C. L. Hagopian [**4**, Theorem 4] generalized this result in two ways in the following:

THEOREM. *If X is a δ-connected continuum, then for any connected nowhere dense subset H of X, the interior of $K(H)$ relative to X is void.*

Hagopian then raised the question as to whether his theorem was true for λ-connected continua [**6**]. The main theorem of this paper is to answer that question in the affirmative by proving that the theorem is true, in fact, for ω-connected

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continua. As consequences of this theorem, several results concerning monotone, upper-semicontinuous decompositions for continua with certain separation properties are established. Hagopian has also shown [**5**, Theorem 2] the equivalence of δ-connectedness and λ-connectedness for plane continua. A short proof is given at the close of this paper showing that in fact ω-connectedness, λ-connectedness and δ-connectedness are all equivalent for plane continua.

THEOREM 1. *If X is an ω-connected continuum, then for each connected nowhere dense subset H of X, the interior of $K(H)$ relative to X is void.*

PROOF. Let H be a connected nowhere dense subset of X and assume that $K(H)$ has a nonvoid interior relative to X. Let W be a nonvoid open subset of X whose closure is contained in the interior of $K(H)\setminus H$. Define T to be the component of $X\setminus W$ that contains H and let $K = T \cap \mathrm{Bd}(W)$. There exist open sets U_0, C_0 such that $\mathrm{diam}(U_0) < 1$, $K \subset C_0 \subset \mathrm{Cl}(C_0) \subset \mathrm{Int}(K(H)\setminus H)$, $\mathrm{dist}(x, K) < 1$ for all $x \in C_0$, $\mathrm{Cl}(U_0) \subset W$ and $\mathrm{Cl}(U_0) \cap \mathrm{Cl}(C_0) = \varnothing$.

Observe that the component P of $X\setminus U_0$ that contains H (and hence contains T) cannot contain C_0. To see this, suppose $C_0 \subset P$. Since $H \not\subset \mathrm{Int}(P)$ (for otherwise $U_0 \not\subset K(H)$), there exists a sequence of points $x_1, x_2, \ldots$ in $X\setminus(P \cup U_0)$ converging to a point of H. For each positive integer i, define S_i to be the x_i-component of $X\setminus U_0$. Then $S = \limsup S_i$ is a continuum in $X\setminus U_0$ that intersects both H and $\mathrm{Cl}(U_0)$. Since $C_0 \subset P$, for each positive integer i, $S_i \cap C_0 \neq \varnothing$, hence $S \cap C_0 = \varnothing$. But $K \subset C_0$ and $S \cap T = \varnothing$ so $S \cap W = \varnothing$. This means that $S \subset T$ and hence that $T \cap \mathrm{Cl}(U_0) \neq \varnothing$ since $S \cap \mathrm{Cl}(U_0) \neq \varnothing$. This is a contradiction since $T \subset X\setminus W$ and $\mathrm{Cl}(U_0) \subset W$. Therefore C_0 is not a subset of P.

Thus we have $X\setminus U_0 = L_0 \cup R_0$, a separation, such that $K \subset P \subset L_0$ and $R_0 \cap C_0 \neq \varnothing$. Define V_0, C_1 to be open sets such that $\mathrm{diam}(V_0) < 1$, $K \subset C_1 \subset \mathrm{Cl}(C_1) \subset L_0 \cap C_0$ and $\mathrm{Cl}(V_0) \subset R_0 \cap C_0$. Now replace U_0 and C_0 by V_0 and C_1, respectively, and follow the argument of the last paragraph word for word down to the final two sentences. This gives a contradiction since $T \subset L_0 \cap (X\setminus W)$ and $\mathrm{Cl}(V_0) \subset R_0$. Therefore C_1 is not a subset of P, the component of $X\setminus V_0$ that contains H. Therefore $X\setminus V_0 = L_1 \cup R_1$, a separation, such that $K \subset L_1 \cap C_1$ and $R_1 \cap C_1 \neq \varnothing$. Note that $L_1 \cap U_0 \neq \varnothing$ and $R_1 \cap U_0 \neq \varnothing$.

We proceed by induction. Assume that open sets U_i, V_i, C_{2i}, C_{2i+1} of X have been defined for $0 \leqslant i \leqslant n-1$ such that

(1) $\mathrm{diam}(U_i) < 1/2^i$, $\mathrm{diam}(V_i) < 1/2^i$;

(2) $X\setminus U_i = L_{2i} \cup R_{2i}$, a separation;

(3) $K \subset C_{2i+1} \subset \mathrm{Cl}(C_{2i+1}) \subset L_{2i} \cap C_{2i}$, $\mathrm{Cl}(V_i) \subset R_{2i} \cap C_{2i}$, $\mathrm{dist}(x, K) < 1/2^i$ for all $x \in C_{2i}$;

(4) $X\setminus V_i = L_{2i+1} \cup R_{2i+1}$, a separation;

(5) $K \subset L_{2i+1} \cap C_{2i+1}$, $R_{2i+1} \cap C_{2i+1} \neq \varnothing$; and

(6) $\mathrm{Cl}(U_{i+1}) \subset U_i \cap R_{2i+1}$ for $i < n-1$.

Let C_{2n} be an open set such that $K \subset C_{2n} \subset \mathrm{Cl}(C_{2n}) \subset L_{2n-1} \cap C_{2n-1}$ and $\mathrm{dist}(x, K) < 1/2^n$ for all $x \in C_{2n}$. By (2)–(5), $R_{2n-1} \cap U_{n-1} \neq \varnothing$, so let U_n be an open set such that $\mathrm{Cl}(U_n) \subset R_{2n-1} \cap U_{n-1}$ and $\mathrm{diam}(U_n) < 1/2^n$. Our previous

argument yields the fact that $X \setminus U_n = L_{2n} \cup R_{2n}$, a separation, such that $K \subset L_{2n} \cap C_{2n}$ and $R_{2n} \cap C_{2n} \neq \varnothing$. Define V_n, C_{2n+1} to be open sets such that $K \subset C_{2n+1} \subset \mathrm{Cl}(C_{2n+1}) \subset L_{2n} \cap C_{2n}$ and $\mathrm{Cl}(V_n) \subset R_{2n} \cap C_{2n}$. As before we have $X \setminus V_n = L_{2n+1} \cup R_{2n+1}$, a separation, such that $K \subset L_{2n+1} \cap C_{2n+1}$ and $R_{2n+1} \cap C_{2n+1} \neq \varnothing$.

Let $\{x\} = \bigcap U_i$ and (taking subsequences if necessary) let $y = \lim V_i$. Note that $y \in K$ since $K = \bigcap C_i$ and $V_i \subset C_{2i}$ for $i = 0, 1, \ldots$. By hypothesis there exists an irreducible subcontinuum I of X from x to y that is decomposable. Set $I = I_x \cup I_y$, where I_x and I_y are continua such that $x \in I_x \setminus I_y$, $y \in I_y \setminus I_x$. Let k be an integer such that $U_k \cap I_y = \varnothing$ and $V_k \cap I_x = \varnothing$. Then $I_x \subset R_{2k+1}$ since $x \in R_{2k+1}$. Also $I_y \subset L_{2k}$ since $y \in K$ and $K \subset \bigcap L_i$. Then since $I_y \subset L_{2k}$, $V_k \subset R_{2k}$ and $y \in L_{2k+1}$, it follows that $I_y \subset L_{2k+1}$. So we have $I = I_x \cup I_y \subset L_{2k+1} \cup R_{2k+1}$, a separation, with $x \in R_{2k+1}$ and $y \in L_{2k+1}$. This is a contradiction, so the assumption that $K(H)$ has a nonvoid interior is false. This completes the proof of the theorem.

A *θ_n-continuum* (*θ-continuum*) is a continuum such that every subcontinuum separates it into at most n components (a finite number of components). The following theorem has been established for θ_n-continua in [**2**, Theorem 2] and for θ-continua in [**1**, Theorem 2].

THEOREM. *Let X be a θ_n-continuum or a θ-continuum. Then X admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors, and which is unique and minimal with respect to the property that the quotient space is a finite graph if and only if whenever H is a nowhere dense subcontinuum of X, it follows that $K(H)$ is nowhere dense.*

As a consequence of this theorem and Theorem 1, the next result follows immediately.

THEOREM 2. *Let X be a θ_n-continuum or a θ-continuum and let X be ω-connected. Then X admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors and which is unique and minimal with respect to the property that the quotient space is a finite graph.*

Specifically for θ_1-continua (continua for which the complement of every subcontinuum is connected), Theorem 3 provides a generalization of Hagopian's decomposition theorem for δ-connected θ_1-continua [**4**, Theorem 5].

THEOREM 3. *Let X be an ω-connected θ_1-continuum. Then X admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors and which is unique and minimal with respect to the property that the quotient space is a simple closed curve.*

The equivalence of δ-connectedness and ω-connectedness for plane continua is observed as a final result. The proof consists of combining the following two theorems.

THEOREM (HAGOPIAN [**4**, THEOREM 2]). *A plane continuum is δ-connected if and only if it cannot be mapped continuously onto D, the Knaster plane indecomposable continuum with one endpoint.*

THEOREM (KRASINKIEWICZ AND MINC [8, THEOREM 10]). *Let X be a plane continuum which can be mapped continuously onto an indecomposable continuum. Then there exist two points x, $y \in X$ such that X contains exactly one subcontinuum that is irreducible between x and y and this subcontinuum is indecomposable.*

THEOREM 4. *If X is an ω-connected plane continuum then X is δ-connected.*

PROOF. Suppose X is not δ-connected. According to Hagopian's theorem X can be mapped continuously onto the Knaster indecomposable continuum D. Then by the theorem of Krasinkiewicz and Minc, there are two points x, $y \in X$ such that there is only one subcontinuum that is irreducible between x and y and it is indecomposable. But this is a contradiction to the ω-connectedness of X. Hence X is δ-connected.

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