ON COLLECTIONWISE NORMALITY
OF PRODUCT SPACES. I
KEIKO CHIBA

ABSTRACT. In this paper the following result will be obtained: Let X be a collectionwise normal \( \Sigma \)-space (in the sense of Nagami [9]) and Y a paracompact first countable \( P \)-space (in the sense of Morita [8]). Then \( X \times Y \) is collectionwise normal.

1. Introduction. Throughout this paper all spaces are Hausdorff spaces.
On collectionwise normality of a product space \( X \times Y \), the following theorems are known.

(I) (KOMBAROV [5]). Let \( X \) be a normal countably compact space and \( Y \) a paracompact sequential space. Then \( X \times Y \) is collectionwise normal.

(II) (YAJIMA [14]). Let \( X \) be a collectionwise normal space which has a \( \sigma \)-closure preserving closed cover by countably compact sets and \( Y \) a paracompact first countable space. Then \( X \times Y \) is collectionwise normal.

We shall consider another condition of \( X \) and \( Y \) such that \( X \times Y \) is collectionwise normal. The following are known.

(III) (NAGAMI [9]). Let \( X \) be a paracompact \( \Sigma \)-space and \( Y \) a paracompact \( P \)-space. Then \( X \times Y \) is paracompact.

(IV) [2]. Let \( X \) be a normal \( M \)-space and \( Y \) a paracompact first countable \( P \)-space. Then \( X \times Y \) is normal.

(V) [2]. There exist a normal \( \sigma \)-space \( X \) and a compact first countable space \( Y \) such that \( X \times Y \) is not normal (see \( \S \)3, Example 3).

In this paper we shall prove the following theorem which contains (IV).

THEOREM. Let \( X \) be a collectionwise normal \( \Sigma \)-space and \( Y \) a paracompact first countable \( P \)-space. Then \( X \times Y \) is collectionwise normal.

The definitions of \( \Sigma \)-spaces are due to Nagami [9], \( P \)-spaces and \( M \)-spaces are due to Morita [8], and \( \sigma \)-spaces are due to Okuyama [10].

2. Proof of Theorem. For the proof, we shall use the following facts.

FACT 1. Let \( \mathcal{A} = \{ A_\gamma \mid \gamma \in \Gamma \} \) be a discrete collection of closed subsets of \( X \). If there exists a normal open cover of \( X \) each of whose members meets at most
one $A_\gamma$, then there are open sets $H_\gamma$ of $X$ such that $H_\gamma \supset A_\gamma$ for each $\gamma \in \Gamma$ and $H_\gamma \cap H_\mu = \emptyset$ if $\gamma \neq \mu$.

Fact 1 is well known.

**FACT 2** [9, Lemma 1.4]. Let $X$ be a $\Sigma$-space. Then $X$ has a $\Sigma$-net $\{\tilde{S}_n| n = 1, 2, \ldots\}$ which satisfies the following conditions:

1. $(N_1)$ $\tilde{S}_n = \{F(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Xi\}$.
2. $(N_2)$ Every $F(\alpha_1, \ldots, \alpha_n) = \bigcup \{F(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) | \alpha_{n+1} \in \Xi\}$.
3. $(N_3)$ For every $x \in X$, there exists a sequence $\alpha_1, \alpha_2, \ldots$ such that $\{F(\alpha_1, \ldots, \alpha_n)| n = 1, 2, \ldots\}$ is a net of $C(x)$.

Here $C(x) = \bigcap \{C(x, \tilde{S}_n)| n = 1, 2, \ldots\}$, $C(x, \tilde{S}_n) = \bigcap \{F| x \in F \in \tilde{S}_n\}$.

**PROOF OF THEOREM.** This proof is a modification of that of (III) (Theorem 4.1 in [9]). Let $X$ be a collectionwise normal $\Sigma$-space and $Y$ a paracompact first countable $P$-space. Let $\{\tilde{S}_n| n = 1, 2, \ldots\}$ be a $\Sigma$-net of $X$ satisfying the conditions $(N_1)$–$(N_3)$ in Fact 2. Since $\tilde{S}_n$ is a locally finite closed cover of $X$ and $X$ is strongly normal, by Katětov [4], there exists a locally finite cozero-set cover $\delta_n = \{H(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Xi\}$ such that

$$F(\alpha_1, \ldots, \alpha_n) \subset H(\alpha_1, \ldots, \alpha_n) \quad \text{for each } \alpha_1, \ldots, \alpha_n \in \Xi.$$

Let $\mathfrak{A}$ be a discrete family of closed subsets of $X \times Y$. Let $\mathfrak{M}(\alpha_1, \ldots, \alpha_n) = \{U_\lambda \times V_\lambda \ (\neq \emptyset) | \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\}$ be the maximal collection satisfying the following conditions:

1. Each $U_\lambda$ is a finite union of cozero-sets $\{U_{\lambda, j}| 1 \leq j \leq m(\lambda)\}$ of $X$ such that $F(\alpha_1, \ldots, \alpha_n) \subset U_\lambda \subset H(\alpha_1, \ldots, \alpha_n)$.

2. Each $V_\lambda$ is an open set of $Y$.

3. Each member of $\mathfrak{M}(\alpha_1, \ldots, \alpha_n) = \{U_{\lambda, j} \times V_\lambda| 1 \leq j \leq m(\lambda)\}$ meets at most one member of $\mathfrak{A}$.

Let us put $V(\alpha_1, \ldots, \alpha_n) = \bigcup \{V_\lambda| \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\}$. Then $V(\alpha_1, \ldots, \alpha_n) \supset V(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Xi$. Since $Y$ is a $P$-space, for each $\alpha_1, \ldots, \alpha_n \in \Xi$, there exists a closed set $K(\alpha_1, \ldots, \alpha_n)$ of $Y$ such that

4. $K(\alpha_1, \ldots, \alpha_n) \subset V(\alpha_1, \ldots, \alpha_n)$.

5. If $\bigcup_{n=1}^{\omega} V(\alpha_1, \ldots, \alpha_n) = Y$, then $\bigcup_{n=1}^{\omega} K(\alpha_1, \ldots, \alpha_n) = Y$, where $\omega$ denotes the first infinite ordinal.

Since $Y$ is paracompact, for each $\alpha_1, \ldots, \alpha_n \in \Xi$, there exists a locally finite collection $\{V_\lambda| \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\}$ of cozero-sets in $Y$ such that:

6. Each $V_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)$.

7. $K(\alpha_1, \ldots, \alpha_n) \subset \bigcup \{V_\lambda| \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\}$.

Let us put $\mathfrak{G}_n = \{U_{\lambda, j} \times V_\lambda| \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n), \alpha_1, \ldots, \alpha_n \in \Xi, 1 \leq j \leq m(\lambda)\}$ for each $n = 1, 2, \ldots$, and put $\mathfrak{G} = \bigcup \{\mathfrak{G}_n| n = 1, 2, \ldots\}$.

Then we have:

8. Each $\mathfrak{G}_n$ is locally finite in $X \times Y$.

9. Each member of $\mathfrak{G}$ meets at most one member of $\mathfrak{A}$.

10. Each member of $\mathfrak{G}$ is a cozero-set in $X \times Y$.

11. $\mathfrak{G}$ is a cover of $X \times Y$.

(8)–(10) are clear.

\[\text{The author first proved this theorem by another method. Y. Yajima pointed out that we can give a simpler proof by modifying the proof of (III).}\]
PROOF OF (11). Let \((x, y) \in X \times Y\) be an arbitrary element. Let \(\alpha_1, \ldots, \alpha_n, \ldots \in \Xi\) be elements such that \(\{F(\alpha_1, \ldots, \alpha_n)|n = 1, 2, \ldots\}\) is a net of \(C(x)\). Then we have \(\bigcup_{n=1}^{\omega} V(\alpha_1, \ldots, \alpha_n) = Y\). To show this, let \(y'\) be an arbitrary element of \(Y\). Then, since \(C(x)\) is countably compact \([9]\) and the family \(\mathfrak{A}\) is discrete, \(\{A_x \in \mathfrak{A}|(C(x) \times \{y'\}) \cap A_x \neq \emptyset\}\) is finite. Therefore, by using the first countability of \(Y\), there is a finite family \(\{M_j|j = 1, 2, \ldots, k\}\) of open sets in \(X\) and an open set \(G\) in \(Y\) such that:

\[(12) \quad C(x) \subset \bigcup_{j=1}^{k} M_j, \quad y' \in G.\]

(13) Each \(M_j \times G\) meets at one member of \(\mathfrak{A}\).

By Lemma 2.1 in \([14]\), there are cozero-sets \(M_j'\) in \(X\) such that \(M_j' \subset M_j\) and \(C(x) \subset \bigcup_{j=1}^{k} M_j'\). Then \(F(\alpha_1, \ldots, \alpha_i) \subset \bigcup_{j=1}^{k} M_j'\) for some \(i\). Let us put \(U_j = M_j' \cap H(\alpha_1, \ldots, \alpha_i)\). Then \(U_j\) are cozero-sets in \(X\) and \((\bigcup_{j=1}^{k} U_j) \times G \in \mathfrak{M}(\alpha_1, \ldots, \alpha_i)\) by the maximality of \(\mathfrak{M}(\alpha_1, \ldots, \alpha_i)\). Thus \(y' \in V(\alpha_1, \ldots, \alpha_i)\).

Therefore we have \(\bigcup_{n=1}^{\omega} K(\alpha_1, \ldots, \alpha_n) = Y\) by (5). Hence \(y \in K(\alpha_1, \ldots, \alpha_n)\) for some \(n\). By (7), \(y \in V_{\lambda}^\prime\) for some \(\lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\). Then

\[(x, y) \in C(x) \times \{y\} \subset F(\alpha_1, \ldots, \alpha_n) \times V_{\lambda}^\prime \subset U_\lambda \times V_{\lambda}^\prime.\]

Since \(x \in U_{\lambda,j}\) for some \(j \leq m(\lambda)\), \((x, y) \in U_{\lambda,j} \times V_{\lambda}^\prime \in \mathfrak{G}_n \subset \mathfrak{G}\).

By (8)–(11), \(\mathfrak{G}\) is a normal open cover of \(X \times Y\), each of whose members meets at most one element of \(\mathfrak{A}\). By Fact 1, there exists a disjoint family \(\{H_A|A \in \mathfrak{A}\}\) of open sets in \(X \times Y\) such that \(H_A \supset A\) for each \(A \in \mathfrak{A}\). Hence \(X \times Y\) is collectionwise normal. The proof of the Theorem is complete.

3. Remarks and examples.

REMARK 1. Our Theorem is neither contained in (I) nor (II) in §1. In fact, let \(X\) be the space of irrationals of \(R\) with the euclidean topology where \(R\) is the real line and \(Y\) the Michael line \([6]\); then \(Y\) is a paracompact first countable space and \(X \times Y\) is not normal \([6]\). Therefore \(X\) does not satisfy the condition in (II). Also \(X\) is not countably compact. But \(X\) is a collectionwise normal \(\Sigma\)-space.

Moreover, this example shows that the condition "\(Y\) is a \(P\)-space" cannot be dropped in the Theorem.

REMARK 2. We cannot weaken the condition "\(Y\) is first countable" to the condition "for each \(y \in Y\) is a \(G_\delta\)-set". In fact the following example exists.

EXAMPLE 1 \([2]\). Let \(X = [0, \omega_1) = \{\alpha|\alpha < \omega_1\}\) with the order topology where \(\omega_1\) is the first uncountable ordinal. Then it is well known that \(X\) is a normal countably compact space. Let \(Y = ([0, \omega) \times [0, \omega_1)) \cup \{(\omega, \omega_1)\}\) with the topology as follows: \(\{[\alpha, \omega] \times [\beta, \omega_1)|\alpha < \omega, \beta < \omega_1\}\) is a neighborhood base of \((\omega, \omega_1)\) and for each \(y \in Y - \{(\omega, \omega_1)\}\), \(y\) is an isolated point of \(Y\). Then \(Y\) is a paracompact perfectly normal space but \(Y\) is not first countable. Also \(X \times Y\) is not normal \([2]\).

REMARK 3. The paracompactness of \(Y\) cannot be weakened to the condition "collectionwise normal". In fact the following example exists.

EXAMPLE 2. There exists a compact space \(X\) and a collectionwise normal perfectly normal first countable space \(Y\) such that \(X \times Y\) is not normal. Let \(Y\) be the space constructed by R. Pol in \([11]\). Then \(Y\) has the above properties, but \(Y\) is not paracompact. Therefore, by the Theorem of Tamano \([13]\), there exists a compact space \(X\) such that \(X \times Y\) is not normal.
REMARK 4. The condition "$X$ is a $\Sigma$-space" cannot be replaced by the condition "$X$ is a $P$-space". In fact, let $X$ be the Sorgenfrey line $[12]$; then $X$ is a paracompact first countable $P$-space such that $X^2 = X \times X$ is not normal.

EXAMPLE 3 [2]. There exists a normal $\sigma$-space $X$ and a compact first countable space $Y$ such that $X \times Y$ is not normal. Let $Y$ be the "two arrow space" i.e., let $E$ be the unit square with lexicographic order (cf. [7, Example 10.4]). Let $Y = (\{y\mid 0 < y \leq 1\} \times \{0\}) \cup (\{y\mid 0 \leq y < 1\} \times \{1\})$ with the subspace topology of $E$. Then $Y$ is a compact first countable space. Let $X$ be the space in Bing's Example H [1] constructed by a suitable set $P$. Then $X$ is a normal $\sigma$-space and $X \times Y$ is not normal (this follows from the proof of Theorem 1 in [3] because $Y$ is separable and not metrizable; also see p. 6 in [2]).

REMARK 5. This author does not know whether we can generalize the condition "$Y$ is first countable" to "$Y$ is sequential" or not.

REFERENCES


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SHIZUOKA UNIVERSITY, OHYA, SHIZUOKA 422, JAPAN