ON SOME CONTINUED FRACTION IDENTITIES OF SRINIVASA RAMANUJAN

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Abstract. The main purpose of this note is to state and prove, in a simple, unified manner, several q-continued fraction expansions found in Ramanujan's "lost" notebook. This is related to some recent works of G. E. Andrews and M. D. Hirschhorn.

0. Introduction. The following continued fraction identities (1)_R-(3)_R and (1)_R found in the "lost" notebook of Ramanujan (terminology due to G. E. Andrews [1]) contain as special cases many of his other identities:

\[
G(0, \lambda q, b, q) = \frac{1 + \lambda q}{1 + \lambda q^2} \frac{bq + \lambda q^2}{1 + \lambda q^2 + \lambda q^2 + \cdots} = \frac{1 + \lambda q}{1 + \lambda q + bq^2 + \cdots} \frac{bq^2 + \lambda q^2 + \cdots}{1 + bq^2 + \cdots}
\]

(2)_R

\[
= \frac{1 + \lambda q}{1 + \lambda q + bq^2 + \cdots} \frac{bq^2 + \lambda q^2 + \cdots}{1 + bq^2 + \cdots}
\]

(3)_R

\[
= \frac{1 + \lambda q}{1 + \lambda q + bq^2 + \cdots} \frac{bq^2 + \lambda q^2 + \cdots}{1 + bq^2 + \cdots}
\]

and, more generally

\[
G(aq, \lambda q, b, q) = \frac{1 + \lambda q}{1 + \lambda q^2} \frac{bq + \lambda q^2}{1 + \lambda q^2 + \lambda q^2 + \cdots} = \frac{1 + \lambda q}{1 + \lambda q + bq^2 + \cdots} \frac{bq^2 + \lambda q^2 + \cdots}{1 + bq^2 + \cdots}
\]

where

\[
G(a, \lambda, b, q) = \sum_{n=0}^{\infty} \frac{q^{n+1/2}(-\lambda/a)_n a^n}{(q)_n (-bq)_n}
\]

Here and in what follows,

\[
(c)_k = \begin{cases} 1 & \text{if } k = 0, \\ (1 - c)(1 - cq) \cdots (1 - cq^{k-1}) & \text{if } k > 0. \end{cases}
\]

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The suffix $R$ signifies that the identity is due to Ramanujan. It is easily seen that $(1)_R - (3)_R$ are themselves special cases, respectively, of $(I)_R$ above and $(II)$ and $(III)_H$ below:

$$(II) \quad G(aq, \lambda q, b, q) = \frac{1}{1 + \frac{aq + \lambda q}{1 - aq + bq + \cdots}} \frac{aq + \lambda q^n}{1 - aq + bq^n + \cdots}$$

$$(III)_H \quad = \frac{1}{1 - b + aq + \frac{b + \lambda q}{1 - b + aq + bq^2 + \cdots}} \frac{b + \lambda q^n}{1 - b + aq^{n+1} + \cdots}$$

The suffix $H$ in $(III)_H$ signifies that the identity is due to M. D. Hirschhorn [3]. Identity $(I)_R$, and thereby $(1)_R$, has been proved independently by Andrews [1] and by Hirschhorn [4]. Andrews has employed $G$ and some auxiliary functions and a transformation of E. Heine; and Hirschhorn has proved it by obtaining a closed form for the $n$th convergent. While Andrews [2] has given a separate proof of the "slightly tricky" identity $(2)_R$, he has extracted $(3)_R$ as a particular case of $(III)_H$ which Hirschhorn [3] has proved by finding a closed form for the $n$th convergent. Many other identities of Ramanujan also follow as pointed out by Andrews and Hirschhorn. However, we have not come across $(II)$ nor a proof of the following Ramanujan identities $(IV)_R$ and $(5)_R$ also listed in the "lost" notebook:

$$(IV)_R \quad G(aq, \lambda q, b, q) = \frac{1}{1 + aq + \frac{\lambda q - abq^2}{1 + q(aq + b)}} \frac{\lambda q^n - abq^{2n}}{1 + q^n(aq + b) + \cdots}$$

$$(5)_R \quad \frac{1 - a + c - a + b + cq - \cdots}{c - b + a + c - a + c/a + \cdots} \frac{bc}{c - b + a + c - b + a/q + \cdots}$$

In what follows we employ, as auxiliary function instead of $G(a, \lambda, b, q)$ a multiple of it, namely,

$$(4^*) \quad g(a, \lambda, b, q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n}{(q)_n(-bq)_n}$$

and will thereby be able to give a simple, unified and self-contained approach to proving $(I)_R$, $(II)$, $(III)_H$, $(IV)_R$ and $(5)_R$. We may observe that in all the identities (I)–(IV) and in $(1)_R - (3)_R$ we may replace $G$ by $g$. We deduce $(I)$–(IV) directly (§§2–5) from three easily proved canonical functional relations (6)–(8) for $g$ (§1) and extract $(5)_R$ (§6) as a particular case of the identity $(II) = (IV)_R$ with $\lambda = 0$.

1. Three canonical functional relations satisfied by $g$.

**Lemma 1 (Key Lemma).** If $|q| < 1$, then $g$ satisfies the following functional relations:

$$(6) \quad g(a, \lambda, b, q) - g(aq, \lambda, b, q) = aqg(aq, \lambda q, bq, q),$$

$$(7) \quad g(a, \lambda, b, q) - g(a, \lambda q, b, q) = \lambda bqg(aq, \lambda q^2, bq, q),$$

$$(8) \quad g(a, \lambda, b, q) - g(a, \lambda, bq, q) = bqg(aq, \lambda q, bq, q).$$

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Proof. Since,
\[ (-\lambda/a)_n - q^n(-\lambda/aq)_n = \begin{cases} 0 & \text{if } n = 0, \\
(-\lambda/a)_{n-1}(1 - q^n) & \text{if } n > 0,
\end{cases} \]
as easily verified, we have
\[ \text{left side of (6)} = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}a^n}{(q)_n(-bq)_n} \left( (-\lambda/a)_n - q^n \left( -\frac{\lambda}{aq} \right)_n \right) \]
\[ = aq(-bq^2)_\infty \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}(aq)^{n-1}(-\lambda/a)_{n-1}}{(q)_{n-1}(-bq^2)_{n-1}} \]
\[ = aqg(aq, \lambda q, bq, q), \text{ proving (6)}. \]
In the penultimate step we have used the obvious identity
\[ \frac{(-bq)_\infty}{(-bq^2)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_{n-1}}. \]
Relations (7) and (8) follow in exactly the same way using
\[ (-\lambda/a)_n - (-\lambda q/a)_n = \begin{cases} 0 & \text{if } n = 0, \\
\frac{\lambda}{a}(-\lambda q/a)_{n-1}(1 - q^n) & \text{if } n > 0,
\end{cases} \]
and
\[ \frac{(-bq)_\infty}{(-bq^2)_n} - \frac{(-bq^2)_\infty}{(-bq^2)_{n-1}} = \frac{(-bq^2)_\infty}{(-bq^2)_n} bq^{n+1}. \]
Lemmas 2–5 proved below are simple combinations of relations (6)–(8). Also, Theorems 1–4 follow directly from Lemmas 2–5, respectively, in a simple manner.

2. Proof of the Ramanujan identity (I)_R.

Lemma 2. \( g \) satisfies
\[ g(a, \lambda, b, q) = g(aq, \lambda q, b, q) + (aq + \lambda q)g(aq, \lambda q^2, bq, q), \]
\[ g(a, \lambda, b, q) = g(a, \lambda q, bq, q) + (bq + \lambda q)g(a, \lambda q^2, bq, q). \]
Proof. Changing \( \lambda \) to \( \lambda q \) in (6) and adding to (7) gives (9), while changing \( \lambda \) to \( \lambda q \) in (8) and adding to (7) gives (10).

Theorem 1. If \( |q| < 1 \), then
\[ (I)_R \]
\[ \rho = \frac{1}{1 + \frac{aq + \lambda q}{1} + \frac{bq + \lambda q^2}{1} + \cdots} \]
\[ = \frac{G(aq, \lambda q, b, q)}{G(a, \lambda, b, q)} = g(aq, \lambda q, b, q) \quad g(a, \lambda, b, q). \]
Proof. Changing \( a \) to \( aq^n \), \( \lambda \) to \( \lambda q^{2n} \), \( b \) to \( bq^n \) in (9) and changing \( a \) to \( aq^{n+1} \), \( \lambda \) to \( \lambda q^{2n+1} \) and \( b \) to \( bq^n \) in (10) we can write (9) and (10), respectively, as

\[
Q_n = \frac{g(aq^n, \lambda q^{2n}, bq^n, q)}{g(aq^{n+1}, \lambda q^{2n+1}, bq^n, q)} = 1 + \frac{aq^{n+1} + \lambda q^{2n+1}}{Q_n},
\]

\[
Q'_n = \frac{g(aq^{n+1}, \lambda q^{2n+1}, bq^n, q)}{g(aq^{n+1}, \lambda q^{2n+2}, bq^{n+1}, q)} = 1 + \frac{bq^{n+1} + \lambda q^{2n+2}}{Q_{n+1}}.
\]

Iterating the last two identities alternately with \( n = 0, 1, 2, \ldots \), we have \((I)_R\). Convergence of the continued fraction follows since \( Q_n, Q'_n \to 1 \) as \( n \to \infty \) when \(|q| < 1\).

3. Proof of identity \((II)\).

Lemma 3. \( g \) satisfies

\[ g(a, \lambda, b, q) = g(aq, \lambda q, b, q) + (aq + \lambda q)g(aq, \lambda q^2, bq, q), \]

\[ g(aq, \lambda, b, q) = (1 - aq + bq)g(aq, \lambda q, bq, q) + (aq + \lambda q)g(aq, \lambda q^2, bq^2, q). \]

Proof. Changing \( \lambda \) to \( \lambda q \) in (6) and adding it to (7), we have (11). Changing \( \lambda \) to \( \lambda q \) and \( b \) to \( bq \) in (6), \( b \) to \( bq \) in (7), taking the negative of (6) and adding these three equalities to (8), we deduce (12).

Theorem 2. If \(|q| < 1\), then

\[ \rho = \frac{1}{1 + \frac{aq + \lambda q}{1 - aq + bq} + \cdots + \frac{aq + \lambda q^n}{1 - aq + bq^n} + \cdots} \]

where \( \rho \) is as in Theorem 1.

Proof. (11) can be written as

\[ \frac{g(aq, \lambda q, b, q)}{g(a, \lambda, b, q)} = \frac{1}{1 + \frac{aq + \lambda q}{g(aq, \lambda q, b, q)}}. \]

Changing \( \lambda \) to \( \lambda q^{n+1} \) and \( b \) to \( bq^n \), (12) can be written as

\[ S_n = \frac{g(aq, \lambda q^{n+1}, bq^n, q)}{g(aq, \lambda q^{n+2}, bq^{n+1}, q)} = (1 - aq + bq^{n+1}) + \frac{aq + \lambda q^{n+2}}{S_{n+1}}. \]

Iterating this with \( n = 0, 1, 2, \ldots \), and using (13) we have (II). Convergence of (II) follows since \( S_n \to 1 \) as \( n \to \infty \) when \(|q| < 1\).


Lemma 4. \( g \) satisfies

\[ g(a, \lambda, bq, q) = (1 - bq + aq)g(aq, \lambda q, bq, q) + (bq + \lambda q)g(aq^2, \lambda q^2, bq, q). \]
Proof. Changing $a$ to $aq$ in (7), $a$ to $aq$ and $\lambda$ to $\lambda q$ in (8), taking the negative of (8) and adding these three equalities to (6), we deduce (14).

**Theorem 3.** If $|q| < 1$, then

$$(III)_H \quad \rho = \frac{1}{1 - b + aq + 1 - b + aq^2 + \cdots} \frac{b + \lambda q}{1 - b + aq^{n+1} + \cdots}$$

where $\rho$ is as in Theorems 1 and 2.

Proof. Changing $a$ to $aq^n$, $\lambda$ to $\lambda q^n$ and $b$ to $b/q$, (14) can be written as

$$T_n = \frac{g(aq^n, \lambda q^n, b, q)}{g(aq^{n+1}, \lambda q^{n+1}, b, q)} = (1 - b + aq^{n+1}) + \frac{b + \lambda q^{n+1}}{T_{n+1}}.$$

Iterating this with $n = 0, 1, 2, \ldots$, we have $(III)_H$. Convergence of $(III)_H$ follows as in the proof of Theorem 2.

5. **Proof of the Ramanujan identity (IV)$_R$**

**Lemma 5.** $g$ satisfies

$$g(a, \lambda, bq, q) = (1 + aq)g(aq, \lambda q, bq, q) + (\lambda q - abq^3)g(aq^2, \lambda q^2, bq^2, q),$$

$$g(aq, \lambda q, b, q) = (1 + q(aq + b))g(aq^2, \lambda q^2, bq^2, q) + (\lambda q^2 - abq^4)g(aq^3, \lambda q^3, bq^3, q).$$

Proof. Change $a$ to $aq$, $\lambda$ to $\lambda q$, $b$ to $bq$ in (6) and multiply the result by $-bq$; change $a$ to $aq$, $b$ to $bq$ in (7); change the negative of (8) and add all these to (6) to obtain (15). Change $a$ to $aq$, $\lambda$ to $\lambda q$ in (15); change $a$ to $aq$ and $\lambda$ to $\lambda q$ in (8) and add to get (16).

**Theorem 4.** If $|q| < 1$, then

$$(IV)_R \quad \rho = \frac{1}{1 + aq + 1 + q(aq + b) + \cdots} \frac{\lambda q - abq^2}{1 + q^n(aq + b) + \cdots}$$

where $\rho$ is as in Theorems 1–3.

Proof. Changing $b$ to $b/q$, (15) can be written as

$$\frac{g(aq, \lambda q, b, q)}{g(a, \lambda, b, q)} = \frac{1}{1 + aq + \frac{\lambda q - abq^2}{g(aq, \lambda q, b, q)}}.$$

Changing $a, \lambda$ and $b$ to $aq^{n-1}, \lambda q^{n-1}$ and $bq^{n-1}$, respectively, (16) can be written as

$$U_n = \frac{g(aq^n, \lambda q^n, bq^{n-1}, q)}{g(aq^{n+1}, \lambda q^{n+1}, bq^n, q)} = 1 + q^n(aq + b) + \frac{\lambda q^{n+1} - abq^{2n+2}}{U_{n+1}}.$$

Iterating (18) with $n = 1, 2, \ldots$, and using (17) we have $(IV)_R$. Convergence of $(IV)_R$ follows since $U_n \to 1$ as $n \to \infty$ when $|q| < 1$. 
6. Proof of the Ramanujan identity \((5)_R\). The following theorem is a corollary to §§3 and 5. In fact, it is a particular case of the identity \((\text{II}) = (\text{IV})_R\).

**Theorem 5.** If \(|q| < 1\), then

\[
\frac{ab}{a + c - a + b + cq - \cdots} \frac{ab}{a + b + cq^n - \cdots} = \frac{1}{c - b + a + c - b + a/q + \cdots} \frac{bc}{c - b + a/q^n + \cdots}.
\]

**Proof.** Changing \(X\) to 0, \(a\) to \(-b/aq\) and \(b\) to \(c/a\) in \((\text{II}) = (\text{IV})_R\) and taking reciprocal we have

\[
g(-b/aq, 0, c/a, q) = \frac{1 + \frac{-b/a}{1 + (b + cq)/a + \cdots}}{1 + \frac{-b/a}{1 + (b + cq)/a + \cdots}}
\]

\[
= \left(1 - \frac{b}{a}\right) + \frac{bcq/a^2}{1 + q(c - b)/a + \cdots} \frac{bcq^{n-1}/a^2}{1 + q^n(c - b)/a + \cdots}.
\]

Multiplying (19) by \(a\) throughout and adding \(c\) throughout we have

\[
c + \frac{ag(-b/aq, 0, c/a, q)}{g(-b/a, 0, c/a, q)} = a + c + \frac{-ab}{a + b + cq} - \cdots \frac{-ab}{a + b + cq^n + \cdots}
\]

\[
= a + c - b + \frac{bcq}{a + (c - b)q} + \cdots \frac{bcq^{n-1}}{a + (c - b)q^n + \cdots}.
\]

We complete the proof by taking the reciprocal of (20) throughout. In addition to proving \((5)_R\) we have thus obtained that each side of \((5)_R\) equals

\[
\left\{c + \frac{ag(-b/aq, 0, c/a, q)}{g(-b/a, 0, c/a, q)}\right\}^{-1}.
\]

**References**


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