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A NOTE ON MAXIMAL OPERATORS AND REVERSIBLE WEAK TYPE INEQUALITIES

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ABSTRACT. A class of maximal operators is shown to satisfy a weak type inequality and a corresponding converse inequality. The results are applicable to a fractionally iterated Hardy-Littlewood maximal operator.

1. In this paper we are interested in studying a class of maximal operators that satisfy a weak type inequality that is in some sense reversible. Let Mf denote the Hardy-Littlewood maximal operator of a function f in $L^1(\mathbf{R}^n)$. As is well known, for instance see [**1**], for suitable constants we have the weak type inequality

$$|\{x \in \mathbf{R}^n : Mf(x) > \lambda\}| \leq \frac{A_n}{\lambda} \int_{\{|f| > \lambda / A_n\}} |f(x)|\, dx$$

and the converse inequality

$$\frac{B_n}{\lambda} \int_{\{|f| > \lambda / B_n\}} |f(x)|\, dx \leq |\{x \in \mathbf{R}^n : Mf(x) > \lambda\}|.$$

Such a converse inequality naturally indicates how good the corresponding weak type inequality is. However, the motivation for the results in this paper is due to an application found in [**4** and **5**]. Knowing the weak type inequality and its converse for the Hardy-Littlewood maximal operator, an important formula for the Nth iteration of the maximal operator was obtained. That is, M applied to f N times is pointwise comparable to

$$\sup_{x \in Q} \frac{1}{|Q|} \int_0^{|Q|} (f \cdot \chi_Q)^*(t) \frac{\log^{N-1}}{(N-1)!} \left(\frac{|Q|}{t}\right) dt,$$

where $(f \cdot \chi_Q)^*$ is the nonincreasing rearrangement of f restricted to the cube Q. As is shown in Theorem 3, this new operator satisfies a weak type inequality that has a converse. We replace $(1/(N-1)!) \log^{N-1}(|Q|/t)$ by a more general function and ask for weak type inequalities that have a converse result.

Let Φ: $(1, \infty) \to [0, \infty)$ be measurable with $\int_0^1 \Phi(1/t)\, dt = 1$. Let f^* be the nonincreasing rearrangement of f, i.e.,

$$f^*(t) = \inf\{s : |\{|f| > s\}| \leq t\}.$$

We define the Φ-maximal operator $M_\Phi f$ as

$$M_\Phi f(x) = \sup \int_0^1 (f \cdot \chi_Q)^*(|Q|t) \Phi\left(\frac{1}{t}\right) dt,$$

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where the sup is extended over all cubes Q with center x and χ_Q is the characteristic function of Q.

The main results, Theorems 1 and 2, are proven for Φ which is either nonincreasing or nondecreasing, $t\Phi(t)$ nondecreasing, and for which there are positive constants C_1, C_2, $0 < \alpha < 2$, $0 < \beta < 1$, such that $C_1 t^\alpha \geq t\Phi(t) \geq C_2 t^\beta$, for $1 \leq t < \infty$. Equivalent maximal operators where $\Phi(t)$ is nondecreasing have been studied (see [**3**]). For this reason perhaps the more interesting weak type estimates in Theorem 1 occur for Φ nonincreasing. Theorem 2 contains the converse inequalities to Theorem 1. As a natural application, we consider the iterated Hardy-Littlewood maximal operator. Our results in Theorems 1 and 2 allow us to consider the case $\Phi(1/t) = \log^a(e/t)$, where a is a given real number. We list the resulting corollaries as Theorem 3.

2. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a measurable function, let

$$\lambda_f(s) = |\{x\colon\ |f(x)| > s\}|,$$

and

$$f^*(t) = \inf\{s\colon\ \lambda_f(s) \leq t\}.$$

For a reference on the properties of $f^*(t)$ see [**2**]. Let $\Phi\colon\ (1,\infty) \to [0,\infty)$ be a nonincreasing (nondecreasing), continuous function such that $\int_0^1 \Phi(1/t)\,dt = 1$. We also require that $t\Phi(t)$ be nondecreasing and that there be positive constants C_1, C_2, $0 < \alpha < 2$, and $0 < \beta < 1$ with $C_1 t^\alpha \geq t\Phi(t) \geq C_2 t^\beta$, for $1 \leq t < \infty$. We define the Φ-maximal opertor of f as

$$M_\Phi f(x) = \sup_Q \int_0^1 (f \cdot \chi_Q)^*(|Q|t)\Phi\left(\frac{1}{t}\right)\,dt,$$

where the sup is extended over all cubes Q with center x, and χ_Q is the characteristic function of Q.

REMARK 1. If f is not measurably constant, i.e., $|\{x\colon\ |f(x)| = a\}| = 0$, for all $a > 0$, then $M_\Phi f$ can be computed as

$$M_\Phi f(y) = \sup \frac{1}{|Q|}\int_Q |f(x)|\Phi\left[\frac{|Q|}{\rho_{Q,f}(x)}\right]\,dx,$$

where the sup is extended over all cubes Q with center y and

$$\rho_{Q,f}(x) = \inf\{t\colon x \in \{z : |f(z)| \geq (f \cdot \chi_Q)^*(t)\}\}.$$

We use the convention that A, B, and C denote constants depending only upon the dimension n and Φ. Any further dependence will be signified by a subscript.

The following lemma contains the inequalities that we will use to show the operator M satisfies a weak type inequality.

LEMMA 1. *Let $\lambda > 0$ and $Q \subset \mathbf{R}^n$ such that*

$$\lambda \leq \int_0^1 (f \cdot \chi_Q)^*(|Q|t)\Phi\left(\frac{1}{t}\right)\,dt.$$

Then if Φ is nonincreasing we have

$$1 \leq \frac{1}{|Q|}\int_{Q\cap\{|f|>\lambda/2\}} C\frac{|f(x)|}{\lambda}\Phi\left[\left(C\frac{|f(x)|}{\lambda}\right)^\beta\right]\,dx.$$

For Φ nondecreasing and $\eta > 1/(2-\alpha)$, there is a constant $C_\eta > 0$ such that

$$1 \le \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} C_\eta \frac{|f(x)|}{\lambda} \Phi\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx.$$

PROOF. Since Φ is normalized, that is $\int_0^1 \Phi(1/t)\,dt = 1$, we have

$$\frac{1}{2} \le \int_{[0,1] \cap \{t : (f \cdot \chi_Q)^*(|Q|t) > \lambda/2\}} \frac{(f \cdot \chi_Q)^*}{\lambda}(|Q|t) \Phi\left(\frac{1}{t}\right) dt.$$

First we do the case where Φ is nondecreasing. For $0 < \delta < 1$ let

$$J_\delta = \left\{ t \in (0,1] : \frac{(f \cdot \chi_Q)^*}{\lambda}(|Q|t) \Phi\left(\frac{1}{t}\right) \le \frac{1}{4} \frac{(1-\delta)}{t^\delta} \right\}.$$

Then

$$\frac{1}{2} \le \int_{\{t : (f \cdot \chi_Q)^*(|Q|t) > \lambda/2\} \setminus J_\delta} \frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda} \Phi\left(\frac{1}{t}\right) dt + \frac{1}{4}|J_\delta|,$$

or

$$\frac{1}{4} \le \int_{\{t : (f \cdot \chi_Q)^*(|Q|t) > \lambda/2\} \setminus J_\delta} \frac{(f \cdot \chi_Q)^*}{\lambda}(|Q|t) \Phi\left(\frac{1}{t}\right) dt.$$

Let $t \in \{t : (f \cdot \chi_Q)^*(|Q|t) > \lambda/2\} \setminus J_\delta$. We use the upper bound on $t\Phi(t)$ to estimate

$$\frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda} \ge \frac{1-\delta}{4\Phi(1/t)t^\delta} \ge \frac{1-\delta}{4C_1} t^{\alpha-(\delta+1)}.$$

We choose δ such that $\alpha - (\delta+1) < 0$, obtaining

$$\left(C_\eta \frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda}\right)^\eta \ge \frac{1}{t},$$

where

$$\eta = \frac{1}{(\delta+1)-\alpha} \quad \text{and} \quad C_\eta = \max\left(4, \left(\frac{4C_1}{1-\delta}\right)\right).$$

Since Φ is nondecreasing we have

$$\begin{aligned} \frac{1}{4} &\le \int_{\{t : (f \cdot \chi_Q)^*(|Q|t) > \lambda/2\}} \frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda} \Phi\left[\left(C_\eta \frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda}\right)^\eta\right] dt \\ &= \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \Phi\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx. \end{aligned}$$

Note that the continuity of $t\Phi[C_\eta t^\eta]$ gives us the above equality.

Now let Φ be nonincreasing. We may assume $C_2 < \frac{1}{2}$, where $t\Phi(t) > C_2 t^\beta$, and

$$\int_{\{(f \cdot \chi_Q)^*(|Q|t) > \lambda/2\}} \frac{(f \cdot \chi_Q)^*(|Q|t)}{\lambda} \Phi\left[\frac{(f \cdot \chi_Q)^*(|Q|t)}{C_2^{2/\beta}\lambda}\right] dt < C_2^{2/\beta}.$$

If not, let $C = C_2^{-2/\beta}$ and we are done, noting that $\beta < 1$ and Φ is nondecreasing. With the above we use the lower bound for $t\Phi(t)$ and the property that $t\Phi(t)$ is nondecreasing to estimate

$$\begin{aligned} C_2 \left[\frac{(f \cdot \chi_Q)^*(|Q|t)}{C_2^{2/\beta}\lambda}\right]^\beta &\le \frac{1}{t} \int_0^t \frac{(f \cdot \chi_Q)^*(|Q|s)}{C_2^{2/\beta}\lambda} \Phi\left[\frac{(f \cdot \chi_Q)^*(|Q|s)}{C_2^{2/\beta}\lambda}\right] ds \\ &\le \frac{1}{t}, \end{aligned}$$

provided $(f\cdot\chi_Q)^*(|Q|t) > \lambda/2$. Thus,

$$\begin{aligned}\frac{1}{2} &\le \int_{\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2\}} \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left(\frac{1}{t}\right)dt \\ &\le \int_{\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2\}} \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left[\left(\frac{(f\cdot\chi_Q)^*(|Q|t)}{C_2\lambda}\right)^{\beta}\right]dt \\ &= \frac{1}{|Q|}\int_{Q\cap\{|f|>\lambda/2\}} \frac{|f(x)|}{\lambda}\Phi\left[\left(\frac{|f(x)|}{C_2\lambda}\right)^{\beta}\right]dx.\end{aligned}$$

This completes the proof of Lemma 1.

Before continuing we would like to make an observation. If a specific nondecreasing Φ was under consideration, a better estimate could be obtained in Lemma 1 by using, for instance, $|t\log^2[2/t]|^{-1}$ instead of $t^{-\delta}$ in the proof. This would result in a sharper, though more complicated, inequality in Theorem 1. Not much is gained if Φ is a logarithmic function.

The next lemma contains the converse inequalities to Lemma 1 that will be used to prove Theorem 2. We follow convention and define

$$\Phi^+(|f(x)|) = \Phi(\max\{1, |f(x)|\}).$$

LEMMA 2. *Let $\lambda > 0$ and $Q \subset \mathbf{R}^n$. If Φ is nonincreasing and $\eta > 1/(2-\alpha)$, then there is a C_η such that*

$$|Q| \le \int_Q \frac{|f(x)|}{\lambda}\Phi^+\left[\left(C_\eta\frac{|f(x)|}{\lambda}\right)^{\eta}\right]dx$$

implies

$$\frac{1}{4} \le \int_0^1 \frac{(f\cdot\chi_Q)^*}{\lambda}(|Q|t)\Phi\left(\frac{1}{t}\right)dt.$$

If Φ is nondecreasing and

$$|Q| \le \int_Q \frac{|f(x)|}{\lambda}\Phi^+\left[\left(\frac{|f(x)|}{\lambda}\right)^{1/(2-\beta)}\right]dx,$$

then there is a constant C such that

$$1 \le C\int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left(\frac{1}{t}\right)dt.$$

PROOF. We first assume Φ is nondecreasing. Let $B = \max\{1, \Phi(1)\}$ and C_2 be the constant, where $t\Phi(t) \ge C_2 t^{\beta}$, $1 \le t < \infty$. We may assume

$$\int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left(\frac{1}{t}\right)dt < \frac{C_2}{2B}.$$

If not, we are done. Using the lower bound for $t\Phi(t)$ we estimate

$$\begin{aligned}\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda} &\le \frac{1}{t\Phi(1/t)}\int_0^t \frac{(f\cdot\chi_Q)^*(|Q|s)}{\lambda}\Phi\left(\frac{1}{s}\right)ds \\ &\le \frac{1}{2Bt^{2-\beta}}.\end{aligned}$$

Thus,

$$1 \le \int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi^+\left[\left(\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{1/(2-\beta)}\right]dt$$

implies

$$\begin{aligned}\frac{1}{2} &\le \int_{\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2B\}}\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left[\left(2B\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{1/(2-\beta)}\right]dt\\ &\le \int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left(\frac{1}{t}\right)dt.\end{aligned}$$

Now let Φ be nonincreasing. We require $C_\eta \ge 2B$. Then

$$1 \le \int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi^+\left[\left(C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{\eta}\right]dt$$

implies

$$\frac{1}{2} \le \int_{\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2B\}}\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left[\left(C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{\eta}\right]dt.$$

For $0<\delta<1$ let

$$J_\delta = \left\{t\in(0,1]: \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left[\left(C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{\eta}\right] \le \frac{(1-\delta)}{4\cdot t^\delta}\right\}.$$

Then

$$\frac{1}{4} \le \int_{\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2B\}\setminus J_\delta}\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left[\left(C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{\eta}\right]dt.$$

For $t\in\{t:(f\cdot\chi_Q)^*(|Q|t)>\lambda/2B\}\setminus J_\delta$ we have, using $C_1t^{-\alpha}\ge t^{-1}\Phi(t^{-1})$ and the above,

$$\frac{C_1}{C_\eta}\left[C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right]^{1-\eta+\alpha\eta} \ge \frac{1}{4}\frac{(1-\delta)}{t^\delta}.$$

Hence, if we require $\eta/(1-\eta+\alpha\eta)>1$, i.e., $\eta>1/(2-\alpha)$, we may choose $\delta=(1-\eta+\alpha\eta)/\eta<1$ and $C_\eta=\max\{2B,4C_1/(1-\delta)\}$ to obtain

$$\left(C_\eta\frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\right)^{\eta} \ge \frac{1}{t}.$$

Thus,

$$\frac{1}{4} \le \int_0^1 \frac{(f\cdot\chi_Q)^*(|Q|t)}{\lambda}\Phi\left(\frac{1}{t}\right)dt$$

and the proof of Lemma 2 is complete.

We now show the maximal $M_\Phi f$ satisfies a weak type inequality. The results are listed as Theorem 1.

THEOREM 1. *Let $\lambda>0$. If Φ is nonincreasing there are constants A and $C>2$ such that*

$$|\{x\in\mathbf{R}^n: M_\Phi f(x)>\lambda\}| \le A\int_{\{|f|>\lambda/2\}}\frac{|f(x)|}{\lambda}\Phi\left[\left(C\frac{|f(x)|}{\lambda}\right)^{\beta}\right]dx.$$

If Φ is nondecreasing there are constants B_η and $C_\eta > 2$ such that, for $\eta > 1/(2-\alpha)$,

$$|\{x \in \mathbf{R}^n : M_\Phi f(x) > \lambda\}| \le B_\eta \int_{\{|f|>\lambda/2\}} \frac{|f(x)|}{\lambda} \Phi\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx.$$

PROOF. We let

$$M_r f(x) = \sup \int_0^1 (f\cdot\chi_Q)^*(|Q|t)\Phi\left(\frac{1}{t}\right) dt,$$

where the sup is extended over all Q with center x and $\operatorname{diam} Q \le r$. It suffices to prove the theorem for $M_r f$ and then let $r \to \infty$.

Let $E_\lambda = \{x : M_r f(x) > \lambda\}$ and $E_{\lambda,R} = E_\lambda \cap \{|x| < R\}$. For $x \in E_{\lambda,R}$, we have a cube Q_x, with center x, and $\operatorname{diam} Q_x \le r$ such that

$$\lambda \le \int_0^1 (f\cdot\chi_Q)^*(|Q|t)\Phi\left(\frac{1}{t}\right) dt.$$

We can now apply the Besicovitch covering theorem [**1**] and select $\{Q_j\} \subset \{Q_x : x \in E_{\lambda,R}\}$ such that $E_{\lambda,R} \subset \bigcup Q_j$ and $\sum \chi_{Q_j} < C$, where C depends only upon n. If we now assume Φ is nondecreasing, then by Lemma 1 we have

$$\begin{aligned} |E_{\lambda,R}| &\le \sum |Q_j| \\ &\le \sum C_\eta \int_{Q_j\cap\{|f|>\lambda/2\}} \frac{|f(x)|}{\lambda}\Phi\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx \qquad \left(\eta > \frac{1}{2-\alpha}\right) \\ &\le B_\eta \int_{\{|f|>\lambda/2\}} \frac{|f(x)|}{\lambda}\Phi\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx. \end{aligned}$$

We now let $R \nearrow \infty$ and then $r \nearrow \infty$ to complete the proof. The case where Φ is nondecreasing uses Lemma 1 and the proof is the same.

We now show that the maximal operator M satisfies an inequality that is related to the converse of the corresponding weak type result of Theorem 1.

THEOREM 2. *Let $\lambda > 0$. If Φ is nonincreasing we have, for some constants A and C_η,*

$$|\{x \in \mathbf{R}^n : M_\Phi f(x) > \lambda A\}| \ge A \int_{\{|f|>\lambda/A\}} \frac{|f(x)|}{\lambda}\Phi^+\left[\left(C_\eta \frac{|f(x)|}{\lambda}\right)^\eta\right] dx,$$

where $\eta > 1/(2-\alpha)$.

If Φ is nondecreasing we have, for some constant B,

$$|\{x \in \mathbf{R}^n : M_\Phi f(x) > \lambda B\}| \ge B \int_{\{|f|>\lambda/B\}} \frac{|f(x)|}{\lambda}\Phi^+\left[\left(\frac{|f(x)|}{\lambda}\right)^{1/(2-\beta)}\right] dx.$$

PROOF. We will just do the nonincreasing case, noting that the nondecreasing case is similar. For each of notation we let $\Phi_1(t) = \Phi[(C_\eta t)^\eta]$ for a fixed $\eta > 1/(2-\alpha)$. Let

$$\overline{f}(x) = \min\{r, |f(x)|\cdot\chi_{B(0,r)}(x)\}.$$

Since $M_\Phi f(x) \ge M_\Phi \overline{f}(x)$, it suffices to prove the theorem for $\overline{f}(x)$ and then let $r \nearrow \infty$.

We apply the Calderon-Zygmund Lemma [**6**] to decompose $\mathbf{R}^n$ into sets E and F where

(i) $F \cap E = \varnothing$,

(ii) $(|\overline{f}(x)|/\lambda)\Phi_1^+(|\overline{f}(x)|/\lambda) \leq 2^n$, a.e. $x \in F$,

(iii) $E = \bigcup Q_j$, where $Q_j \cap Q_k = \varnothing$ for $j \neq k$ and

$$2^n \leq \frac{1}{|Q_j|} \int_{Q_j} \frac{|\overline{f}(x)|}{\lambda} \Phi_1^+ \left(\frac{|\overline{f}(x)|}{\lambda} \right) dx \leq 2^{2n}.$$

We note that (ii) implies $|\overline{f}(x)| \leq 2^n \lambda / \Phi(1)$ for a.e. $x \in F$. For a fixed $x \in Q_j$ let Q_x be the smallest cube centered at x containing Q_j. Then the above implies

$$1 \leq \frac{1}{|Q_x|} \int_{Q_x} \frac{|\overline{f}(x)|}{\lambda} \Phi_1^+ \left(\frac{|\overline{f}(x)|}{\lambda} \right) dx.$$

Hence, by Lemma 2, we have $M_\Phi \overline{f}(x) \geq \lambda/4$ for every $x \in Q_j$. Summing over Q_j and letting $A = \min\{2^{-2n}, 2^{-n}\Phi(1)\}$ completes the proof.

3. Natural and interesting examples arise when Φ is a logarithmic type of function. In such a case Theorems 1 and 2 assume an elegant form since the exponents do not play a significant role.

Consider the following formula for the Nth iteration of the Hardy-Littlewood maximal operator (see [**5**, p. 5]):

$$\underbrace{M \cdots M}_{N \text{ times}} f(x) \sim \sup_{x \in Q} \int_0^1 (f \cdot \chi_Q)^*(|Q|t) \frac{\log^{N-1}(1/t)}{(N-1)!}\, dt.$$

We generalize the above formula for $-\infty < a < \infty$ as follows. For a real let

$$M_a f(x) = \sup_{x \in Q} \int_0^1 (f \cdot \chi_Q)^*(|Q|t) \frac{\log^a(e/t)}{C(a)}\, dt,$$

where $C(a) = \int_0^1 \log^a(e/t)\, dt$. We note that e/t avoids the strong singularity at $t = 1$ if $a \leq -1$. For this maximal operator, Theorems 1 and 2 assume a recognizable form which we state as Theorem 3.

THEOREM 3. *Let $\lambda > 0$ and a be a given fixed number. Then there exist constants A and B which depend only on dimension N and a such that*

$$|\{x \in \mathbf{R}^n : M_a f(x) > \lambda\}| \leq A \int_{\{|f| > \lambda/A\}} \frac{|f(x)|}{\lambda} \left[1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right]^a dx,$$

$$|\{x \in \mathbf{R}^n : M_a f(x) < \lambda\}| \geq B \int_{\{|f| > \lambda/B\}} \frac{|f(x)|}{\lambda} \left[1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right]^a dx.$$

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