C_p-PERTURBATIONS OF NEST ALGEBRAS

GARETH J. KNOWLES

ABSTRACT. Given the ideal C_p and a nest algebra A in \( \mathcal{L}(H) \) there are two corresponding subalgebras of \( \mathcal{L}(H) \). The first consists of all C_p-perturbations of A. The second, a natural generalization of the quasitriangular algebra corresponding to A, consists of all T in \( \mathcal{L}(H) \) with \( E \to (I - E)TE \) continuous from \( \text{Lat} A \) into \( C_p \). Necessary and sufficient conditions are given for these algebras to be identical.

1. Introduction. The notion of a quasitriangular algebra corresponding to some arbitrary nest of projections was first introduced in [3]. In [3], a particularly useful characterization of a quasitriangular algebra as the collection of all compact perturbations of the corresponding nest algebra, is given. From this characterization many useful developments followed [1, 3]. This paper is concerned with studying a class of algebras, the quasitriangular p-classes, that are a natural generalization of a quasitriangular algebra. The main result contained here is that the Arveson characterization fails for \( 1 \leq p < \infty \). Necessary and sufficient conditions are given for the characterization to hold. This research was undertaken while in receipt of a Science Research Council award during the preparation of the authors Ph.D. under the supervision of J. Erdos.

A totally ordered collection of subspaces of some Hilbert space H closed under the taking of suprema and infima is termed a (complete) nest. Given such a nest \( \mathcal{N} \) of subspaces there is a corresponding totally ordered set of projections on the subspaces on \( \mathcal{N} \) which will be denoted by \( \mathcal{E} \). Following [3], the quasitriangular algebra corresponding to such a nest \( \mathcal{N} \) consists of all \( T \in \mathcal{L}(H) \) (the collection of all bounded linear operators acting on H) with the property that \( E \to (I - E)TE \) is a continuous map from \( \mathcal{E} \), with the strong operator topology, to \( C_\infty \), the ideal of all compact operators with the norm topology. The algebra \( \text{Alg} \mathcal{N} \) of all operators acting on \( H \) leaving each subspace in \( \mathcal{N} \) invariant will be denoted by \( A \).

DEFINITION. An operator T in \( \mathcal{L}(H) \) is said to be in the quasitriangular p-class corresponding to \( \mathcal{N} \), denoted by \( T_p (= T_p(\mathcal{N})) \) whenever

(i) \( (I - E)TE \) is in \( C_p \) for each \( E \) in \( \mathcal{E} \) and

(ii) \( E \to (I - E)TE \) is continuous from \( \mathcal{E} \), with the strong operator topology to \( C_p \), with the p-norm.

The proof of the following is based on that of [3] for \( C_\infty \).

LEMMA 1. The algebra \( A + C_p \) is a subalgebra of \( T_p \).

PROOF. Certainly if \( T = A + K \) for some \( A \in A \) and \( K \in C_p \), then \( (I - E)TE \) is in \( C_p \). Suppose that \( 0 \neq K = x \otimes y \) for some \( x, y \) in \( H \). Then

\[
(I - E)KE - (I - F)KF = -Ex \otimes (E - F)y + (E - F)x \otimes (I - F)y.
\]

Received by the editors March 8, 1982 and, in revised form, September 12, 1983. 1980 Mathematics Subject Classification. Primary 47A55.
Thus
\[ ||(I - E)KE - (I - F)KF||_p \leq ||(E - F)y|| \cdot ||x|| + ||(E - F)x|| \cdot ||y|| \]
from which (ii) now follows: Let \( R \) be a finite rank operator. Then by linearity it follows that (ii) holds for \( K = R \). Choose \( R_n \) finite rank with \( ||K - R_n||_p \to 0 \). For each \( E \in \mathcal{E} \),
\[ ||(I - E)KE||_p \leq ||(I - E)(K - R_n)E||_p + ||(I - E)R_nE||_p. \]
Since \( ||(I - E)(K - R_n)E||_p \to 0 \) as \( n \to \infty \) it follows that the map \( E \to (I - E)KE \) is the uniform limit of continuous functions, and hence continuous.

**Corollary 2.** When \( \mathcal{N} \) is a finite nest, then \( \mathcal{A} + \mathcal{C}_p \supseteq \mathcal{T}_p \).

**Proof.** By Lemma 1 it will be sufficient to show \( \mathcal{A} + \mathcal{C}_p \supseteq \mathcal{T}_p \). Let \( \mathcal{N} \) be the nest \( 0 = E_0 < E_1 < \cdots < E_m = I \). For any \( T \in \mathcal{T}_p \)
\[ T = \sum_{n=1}^{m} (I - E_n)T(E_n - E_{n-1}) + \sum_{n=1}^{m} (E_n - E_{n-1})T(I - E_{n-1}), \]
where the first summand is in \( \mathcal{C}_p \) and the second in \( \mathcal{A} \).

**Theorem 3.** If \( \mathcal{N} \) is an infinite nest then \( \mathcal{A} + \mathcal{C}_p \) is strictly contained in \( \mathcal{T}_p \) for \( 1 \leq p < \infty \).

**Proof.** Using the well-known fact that \( \mathcal{E} \) is compact in the strong operator topology, a sequence of distinct projections can be found with either \( E_n \not\in P \) or \( E_n \not\in sP \) for some \( P \) in \( \mathcal{E} \). Let us assume that \( E_n \not\in P \). For each \( n \in N \), choose \( x_n \) a unit vector in the range of \( (E_n - E_{n-1}) \) and define the bounded operator \( K \) in \( \mathcal{L}(H) \) by
\[ K = \sum_{i=1}^{\infty} i^{-1/2} x_i \otimes x_{i+1}. \]
The following will now be shown to hold true for this \( K \):

(i) \( (I - E)KE \) is in \( \mathcal{C}_p \) for each \( E \) in \( \mathcal{E} \); \( 1 \leq p \leq \infty \).

(ii) \( E \to (I - E)KE \) is continuous from \( \mathcal{E} \) with the strong operator topology to \( \mathcal{C}_1 \) with the trace-class norm.

(iii) \( K \notin \mathcal{A} + \mathcal{C}_2 \).

(i) Fix \( E \in \mathcal{E} \) with \( E < P \); then by assumption we can find an \( n \) with \( E < E_n < P \), giving \( (I - E)KE = \sum_{i=1}^{n} i^{-1/2} E x_i \otimes (I - E)x_{i+1} \) is in \( \mathcal{C}_p \) for \( 1 \leq p \leq \infty \).

(ii) Let \( E \in \mathcal{E} \) satisfy \( E < P \). It follows that, for some \( n \in N \), \( E_{n-1} < E \leq E_n \). Given \( \varepsilon > 0 \) choose \( F_1 \) as follows: when \( E \) has an immediate predecessor \( E_- \) let \( F_1 = E_- \); when \( E \) has no immediate predecessor choose \( F_1 \) in \( \mathcal{E} \) with \( E_{n-1} < F_1 < E \) and \( \| (E - F_1)x_j \| < \varepsilon/8 \) for \( j = 1, 2, \ldots, n + 1 \). For an \( F \) with \( F_1 < F \leq E \) it now follows that
\[ (I - E)KE - (I - F)KF \]
\[ = (n - 1)^{-1/2} (Ex_{n-1} \otimes (I - E)x_n - Fx_{n-1} \otimes (I - F)x_n) \]
\[ + n^{-1/2} (Ex_n \otimes (I - E)x_{n+1} - Fx_n \otimes (I - F)x_{n+1}) \]
\[ = (n - 1)^{-1/2} [E x_{n-1} \otimes (E - F)x_n + (E - F)x_{n-1} \otimes (I - F)x_n] \]
\[ + n^{-1/2} [Ex_n \otimes (E - F)x_{n+1} + (E - F)x_n \otimes (I - F)x_{n+1}]. \]
Let $T$ be the first term in the above expansion, that is

$$T = (n-1)^{-1/2}(-Ex_{n-1} \otimes (E-F)x_n).$$

Thus $TT^*$ is a rank one operator with $\sigma((TT^*)^{1/2}) = (n-1)^{-1/2}\|Ex_{n-1}\|$. From this it now follows that $\|T\|_1 < \varepsilon/8$. Similar computations for the other three terms will now show that $\|(I-E)KE - (I-F)KF\|_1 < \varepsilon/2$ for $F_1 < F \leq E$.

Similarly, when $E_k \leq E < E_{k+1}$ for some $k$, given $\varepsilon > 0$, choose $F_2$ as follows: when $E$ has an immediate successor $E_+$ let $F_2$ be $E_+$; when $E$ has no immediate successor choose $F_2$ in $\mathcal{E}$ with $E < F_2 < E_{k+1}$ satisfying

$$\|((F_2 - E)x_j\| < \varepsilon/8 \quad \text{for} \quad j = 1, 2, \ldots, k + 1.$$ 

An exactly similar argument to the one above will show that whenever $G$ in $\mathcal{E}$ satisfies $E \leq G < F_2$ then $\|(I-G)KG - (I-E)KE\|_1 < \varepsilon/2$. Thus, it is concluded that $\|(I-G)K\|_1 < \varepsilon/2$ for $G, F \in (F_1, F_2)$, showing continuity at $E < P$. Note that, for $E \geq E_n$, $\|(I-E)KE\|_1 \leq 2n^{-1/2}$ which implies continuity at $P$. A parallel argument deals with the case that there is a sequence of distinct projections $E_n \searrow sP$ in $\mathcal{E}$.

(iii) First note that $\|K\|^2 = \sum_{j=1}^{\infty} \|(E_{j+1} - E_j)K(E_j - E_{j-1})\|^2 = \sum_{j=1}^{\infty} j^{-1}$, showing $K$ is not in $C_2$. Suppose $\mathcal{F}$ is some arbitrary nest of projections acting on some Hilbert space $H$ and $X$ acting on $H$ is in $C_p$ $(1 < p < \infty)$. It is shown in [2, Theorem 3.2] that if $X$ is rewritten as $X = \mathcal{L}(X) + \mathcal{D}(X) + \mathcal{Z}(X)$, then these define bounded operators, in fact they are also in $C_p$. Applying this to the above, it is first noted that $K = \mathcal{A}(K)$ and $\mathcal{A}(A) = 0$ for any $A$ where we decompose $K$ relative to $\mathcal{E}$. Suppose then that it were possible to write $K$ in the form $A + C$ for some $A \in \mathcal{A}$ and $C \in \mathcal{C}_2$. Therefore,

$$K = \mathcal{A}(K) = \mathcal{A}(K - A) = \mathcal{A}(C)$$

which is in $C_2$, contradicting the above. This completes the argument in the case $p = 2$. It is easily seen that a simple adjustment of the eigenvalues in the definition of $K$ will provide an analogous counterexample for a general $p$, $1 < p < \infty$.

The case for $p = 1$ now follows from that of $p = 2$. Suppose $K$, as above, were of the form $K = A + C$ for some $A$ in Alg $\mathcal{E}$ and $C$ in $\mathcal{C}_1$. Decompose $C$ in $\mathcal{C}_2$ as before. It will now follow that $K$ is in $C_2$, a contradiction.

**DEFINITION.** For each $T \in \mathcal{A} + C_p$ $(\infty > p > 1)$ set

$$\|\|T\|_p = \|A + \mathcal{L}(C) + \mathcal{D}(C)\| + \sup_{E \in \mathcal{E}} \|(I-E)TE\|_p$$

where $T = A + C$ and $C$ decomposes into $\mathcal{L}(C) + \mathcal{D}(C) + \mathcal{A}(C)$ with respect to $\mathcal{E}$. 
Lemma 4. \( ||| \cdot |||_p \) is a norm on \( \mathcal{A} + \mathcal{C}_p \).

Proof. It is obviously well defined, subadditive and satisfies \( ||| \alpha T |||_p = |\alpha| \cdot ||| T |||_p \). Suppose then that \( ||| T |||_p = 0 \). It follows from [1, Theorem 1.1] that \( T \) is in \( \mathcal{A} \). If \( T = A + C \) for some \( A \in \mathcal{A}, C \in \mathcal{C}_p \)

\[
T = A + C = A + \mathcal{L}(C) + \mathcal{D}(C),
\]

from which it follows that \( T = 0 \).

Remarks. It is not very hard to show that \( \mathcal{A} + \mathcal{C}_p \) is not complete in the \( ||| \cdot |||_p \) norm. For example, take a sequence \( \{A_n\}_{n=1}^{\infty} \) where

\[
A_n = \sum_{j=1}^{n} b_j(x_j \otimes x_{j+1}) \quad \text{for some } \{b_n\} \in c_0 \setminus l_p.
\]

It would be of interest to know whether its completion in this norm is \( \overline{T_p} \).

References


Department of Mathematics, Texas Tech University, Lubbock, Texas 79409