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A CHAOTIC FUNCTION POSSESSING A SCRAMBLED SET WITH POSITIVE LEBESGUE MEASURE

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ABSTRACT. A continuous function, chaotic in the sense of Li and Yorke, is constructed which possesses a scrambled set of positive Lebesgue measure.

Introduction. A continuous function $f: I \to I$, where I is a real finite interval, is called chaotic (in the sense of Li and Yorke [**1**]) provided there exists an uncountable set $S \subset I$ such that for any $x, y \in S$, $x \neq y$, and p any periodic point of f:

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0, \tag{1}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0, \tag{2}$$

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0, \tag{3}$$

where f^n is the nth iterate of f. We call any such set S a scrambled set of f. We call a scrambled set E extremally scrambled, iff for any $x, y \in E$, $x \neq y$, and p any periodic point of f:

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = \operatorname{diam}(I), \tag{4}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0, \tag{5}$$

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \geq \operatorname{diam}(I)/2. \tag{6}$$

J. Smítal has shown that $f: [0,1] \to [0,1]$ where

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2 - 2x, & 1/2 < x \leq 1, \end{cases}$$

has an extremally scrambled set of Lebesgue outer measure 1 [**2**]. This set is not Lebesgue measurable, and Smítal's construction of it makes use of the continuum hypothesis.

In the literature there is no known chaotic function possessing a scrambled set of positive Lebesgue measure. Here, without using the continuum hypothesis, we construct a continuous chaotic function $f: [0,1] \to [0,1]$, along with an extremally scrambled set E of Lebesgue measure $\frac{1}{8}$.

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First we consider the class $\mathcal{G}$ of all continuous functions $g\colon [0,1] \to [0,1]$ such that $g(x) = 3x$ for $x \in [0, \frac{1}{3}]$ and $g(x) = 3x - 2$ for $x \in [\frac{2}{3}, 1]$, and find a set $K \subset (0, \frac{1}{3})$ which is an extremally scrambled set of every $g \in \mathcal{G}$. Any such g is clearly chaotic. Then we construct a fat Cantor set $E \subset [\frac{3}{8}, \frac{5}{8}]$ with Lebesgue measure $\frac{1}{8}$. Finally we choose a particular $f \in \mathcal{G}$ which, when restricted to E, is a monomorphism (actually a homeomorphism) from E to K. Since K is an extremally scrambled set of f, E is one also.

Preliminaries. Let N denote the natural numbers. Let $\Omega = \{0,1\}^N$ be the space of all one-sided sequences of two symbols, along with the dictionary ordering relation $<$, and the topology of coordinatewise convergence. The shift σ on Ω is defined by $(\sigma\omega)_k = \omega_{k+1},\ k \in N$.

Let the map $\varphi\colon \Omega \to [0,1]$ be defined by

$$\varphi(\omega) = 2\sum_{k=1}^{\infty} \omega_k 3^{-k}.$$

Then the image under φ of Ω is the usual "middle thirds" Cantor set C, and φ is an order preserving homeomorphism from Ω to C. Also, for any $g \in \mathcal{G}$, we have

$$g(\varphi(\omega)) = 3\varphi(\omega) - 2\omega_1 = 2\sum_{k=1}^{\infty} \omega_{k+1} 3^{-k} = \varphi(\sigma\omega).$$

And, for $n \in N$, $g^n(\varphi(\omega)) = \varphi(\sigma^n\omega)$.

The following simple lemma will prove useful.

LEMMA. (i) *If* $g \in \mathcal{G}$, $\alpha, \beta \in \Omega$, $\alpha_{n+j} = \beta_{n+j}$ *for* $j = 1, \ldots, k$, *then*

$$|g^n(\varphi(\alpha)) - g^n(\varphi(\beta))| \le 3^{-k}.$$

(ii) *If* $g \in \mathcal{G}$, $\alpha \in \Omega$, $\alpha_{n+j} = 0$ $(\alpha_{n+j} = 1)$, *for* $j = 1, \ldots, k$, *then*

$$g^n(\varphi(\alpha)) \le 3^{-k} (g^n(\varphi(\alpha)) \ge 1 - 3^{-k}).$$

PROOF. The lemma clearly follows from the fact that

$$g^n(\varphi(\omega)) = \varphi(\sigma^n\omega) = 2\sum_{j=1}^{\infty} \omega_{n+j} 3^{-j}.$$

I. We construct a set $K \subset C$, which is an extremally scrambled set of every $g \in \mathcal{G}$. The method used here is similar to that used by M. Osikawa and Y. Oono to construct a scrambled set [**3**].

Let $r\colon N \to N$ be defined by

$$r(k) = \inf\left\{ l \in N \mid k \le \sum_{j=1}^{l} j^2 + 2j \right\}$$

and let $s\colon N \to N$ be defined by

$$s(k) = \sup\left\{ l \in N \cup \{0\} \mid l + 1 \le \left(k - \sum_{j=1}^{r(k)-1} j^2 + 2j \right) / r(k) \right\}$$

where $\sum_{j=1}^{0} j^2 + 2j$ is defined to be 0. Then define $Z: \Omega \to \Omega$ by

$$(Z(\omega))_k = \begin{cases} 0 & \text{if } s(k) = 0, \\ 1 & \text{if } s(k) = 1, \\ \omega_{s(k)-1} & \text{if } s(k) \geq 2, \end{cases}$$

i.e., $Z(\omega) = 01\omega_1 0011\omega_1\omega_1\omega_2\omega_2 \cdots$. Let K be the set $\varphi(Z(\Omega))$.

The map $Z: \Omega \to Z(\Omega)$ is an order preserving homeomorphism, and since $\varphi: \Omega \to C$ is also, we see that K is homeomorphic to C. This fact will be important in §III, where we construct a homeomorphism from the fat Cantor set E of §II to K.

PROPOSITION. *K is an extremally scrambled set of every $g \in \mathcal{G}$.*

PROOF. Let $x, y \in K$, $x < y$, and p be any periodic point of g with period q. It suffices to show that equations (4)–(6) hold with g in place of f.

Let $\alpha = \varphi^{-1}(x)$ and $\beta = \varphi^{-1}(y)$. Then by the construction of K we have, for any $k \in N$, infinitely many $n \in N$ such that $\alpha_{n+j} = \beta_{n+j} = 0$ for $j = 1, \ldots, k$. Thus, by (i) of the Lemma, (5) is satisfied. Since $x < y$, we also have infinitely many $n \in N$ such that $\alpha_{n+j} = 0$ and $\beta_{n+j} = 1$, for $j = 1, \ldots, k$. Thus, by (ii) of the Lemma, (4) is satisfied. We also have infinitely many $n \in N$ such that $\alpha_{n+j} = 0$ ($\alpha_{n+j} = 1$) for $j = 1, \ldots, k+q$ so that

$$\limsup_{n\to\infty} |g^n(x) - g^n(p)| \geq \max\{g^n(p) | n \in N\}$$

$$\left(\limsup_{n\to\infty} |g^n(x) - g^n(p)| \geq \max\{1 - g^n(p) | n \in N\}\right)$$

holds. Therefore,

$$\limsup_{n\to\infty} |g^n(x) - g^n(p)| \geq 1/2$$

holds.

II. Here we constructed a fat Cantor set $E \subset [\frac{3}{8}, \frac{5}{8}]$ so that $m(E) = \frac{1}{8}$ [**4**, **5**]. The set E will be the countable intersection of a nested sequence of compact sets $\{E_n\}$ where each E_n is the union of 2^n disjoint closed intervals, exactly two of these intervals being contained in each one of the 2^{n-1} disjoint closed intervals of E_{n-1}. Also, the diameters of the intervals comprising E_n go uniformly to 0 as n goes to ∞.

Let $\Gamma_n = \{0, 1\}^n$ denote the set of all two symbols of length n, and γ_k is the kth coordinate of $\gamma \in \Gamma_n$. There are 2^n elements of Γ_n.

The sine of $\pi/6$ can be written as an infinite product [**6**] as follows,

$$\sin\frac{\pi}{6} = \frac{\pi}{6}\prod_{l=1}^{\infty}\left(1 - \frac{1}{36l^2}\right) = \frac{1}{2}.$$

Let $E_0 = [\frac{3}{8}, \frac{5}{8}]$ and let E_n be the union of the 2^n disjoint closed intervals $I(\gamma)$, where $\gamma \in \Gamma_n$. The right-hand endpoint of $I(\gamma)$ is

$$b(\gamma) = \frac{5}{8} - \frac{1}{4}\sum_{k=1}^{n} \gamma_k 2^{-k}\frac{\pi}{6}\prod_{l=1}^{6}\left(1 - \frac{1}{36l^2}\right)$$

and the left-hand endpoint of $I(\gamma)$ is

$$a(\gamma) = b(\gamma) - \frac{1}{4}2^{-n}\frac{\pi}{6}\prod_{l=1}^{n+1}\left(1 - \frac{1}{36l^2}\right).$$

It follows that if $n \geq 2$, $\gamma \in \Gamma_n$, $\lambda \in \Gamma_{n-1}$, then either $I(\gamma)$ and $I(\lambda)$ are disjoint, or $I(\gamma)$ is contained in $I(\lambda)$. The latter occurs iff $\gamma_k = \lambda_k$ for $k = 1, \ldots, n-1$. The Lebesgue measure of E_n is $\frac{1}{4}\frac{\pi}{6}\prod_{l=1}^{n+1}(1 - 1/36l^2)$ and, since $E_n \downarrow E = \bigcap_n E_n$, we have $m(E) = \lim_{n\to\infty} E_n = \frac{1}{8}$.

A point is in E iff it is a limit point of the right-hand endpoints of the intervals comprising the sets E_n. So, the map $\psi\colon \Omega \to E$ defined by

$$\psi(\omega) = \frac{5}{8} - \frac{1}{4}\sum_{k=1}^{\infty}\omega_k 2^{-k}\frac{\pi}{6}\prod_{l=1}^{k}\left(1 - \frac{1}{36l^2}\right)$$

is an order reversing homeomorphism.

III. We now choose a particular $f \in \mathcal{G}$ so that f restricted to E is a homeomorphism (monomorphism is all that is necessary) from E to K. Then for any $x, y \in E$, $x \neq y$, p any periodic point of f, we have $f(x)$, $f(y) \in K$, $f(x) \neq f(y)$, $f(p)$ a periodic point of f. Thus E will be an extremally scrambled set of f.

Let $f(t) = \varphi(Z(\psi^{-1}(t)))$ for $t \in E$. Since $f \in \mathcal{G}$, we have $f(t) = 3t$ for $t \subset [0, \frac{1}{3}]$ and $f(t) = 3t - 2$ for $t \subset [\frac{2}{3}, 1]$. We now define $f(t)$ for $t \subset E^c \cap (\frac{1}{3}, \frac{2}{3})$ to be a linear interpolation of the values of f on the nearest points of $E \cup \{\frac{1}{3}\} \cup \{\frac{2}{3}\}$ to t. More precisely, for $t \subset E^c \cap (\frac{1}{3}, \frac{2}{3})$, we define

$$f(t) = f(t_l) + \frac{(t - t_l)}{(t_r - t_l)}(f(t_r) - f(t_l))$$

where $t_l = \sup\{t' \in E \mid t' < t\} \cup \{\frac{1}{3}\}$ and $t_r = \inf\{t' \in E \mid t' > t\} \cup \{\frac{2}{3}\}$.

PROPOSITION. *The function f is continuous and K is the homeomorphic image under f of E.*

PROOF. It suffices to show that f is continuous, strictly decreasing on $[\frac{1}{3}, \frac{2}{3}]$, and $f(E) = K$.

The image under f of E is the image under $\varphi \circ Z$ of $\psi^{-1}(E)$, but $\psi^{-1}(E) = \Omega$, so

$$f(E) = \varphi(Z(\psi^{-1}(E))) = \varphi(Z(\Omega)) = K.$$

The function f is continuous and strictly decreasing on $E \cup \{\frac{1}{3}\} \cup \{\frac{2}{3}\}$, since $f(\frac{1}{3}) = 1$, $f(\frac{2}{3}) = 0$, $f(E) = K \subset (0, \frac{1}{3})$, and also ψ^{-1} is continuous and order reversing and both φ and Z are continuous and order preserving. On the rest of $[\frac{1}{3}, \frac{2}{3}]$, f is just contained by linear interpolation, so f is continuous and strictly decreasing on $[\frac{1}{3}, \frac{2}{3}]$.

Conclusion. The f constructed in the third part is a chaotic function possessing an extremally scrambled set E of Lebesgue measure $\frac{1}{8}$.

REMARK. It is easy to find a chaotic function $h\colon [0,1]^r \to [0,1]^r$ which has a scrambled set of positive r dimensional Lebesgue measure m_r. Let h be defined by

$$h(x_1, \ldots, x_r) = (f(x_1), \ldots, f(x_r)).$$

Then E^r is a scrambled set of h and $m_r(E^r) = (\frac{1}{8})^r$. In fact, for any $x, y \in E^r$, $x \neq y$, p any periodic point of h, we have

$$\limsup_{n\to\infty} |h^n(x) - h^n(y)| \geq 1, \tag{7}$$

$$\liminf_{n\to\infty} |h^n(x) - h^n(y)| = 0, \tag{8}$$

$$\limsup_{n\to\infty} |h^n(x) - h^n(p)| \geq \sqrt{r}/2. \tag{9}$$

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