A CHAOTIC FUNCTION POSSESSING A SCRAMBLED SET WITH POSITIVE LEBESGUE MEASURE

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ABSTRACT. A continuous function, chaotic in the sense of Li and Yorke, is constructed which possesses a scrambled set of positive Lebesgue measure.

Introduction. A continuous function \( f : I \to I \), where \( I \) is a real finite interval, is called chaotic (in the sense of Li and Yorke [1]) provided there exists an uncountable set \( S \subset I \) such that for any \( x, y \in S \), \( x \neq y \), and \( p \) any periodic point of \( f \):

\[
\begin{align*}
(1) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0, \\
(2) & \quad \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \\
(3) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(p)| > 0,
\end{align*}
\]

where \( f^n \) is the \( n \)th iterate of \( f \). We call any such set \( S \) a scrambled set of \( f \). We call a scrambled set \( E \) extremally scrambled, iff for any \( x, y \in E \), \( x \neq y \), and \( p \) any periodic point of \( f \):

\[
\begin{align*}
(4) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| = \text{diam}(I), \\
(5) & \quad \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \\
(6) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(p)| \geq \text{diam}(I)/2.
\end{align*}
\]

J. Smítal has shown that \( f : [0, 1] \to [0, 1] \) where

\[
f(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2 - 2x, & 1/2 < x \leq 1, \end{cases}
\]

has an extremally scrambled set of Lebesgue outer measure 1 [2]. This set is not Lebesgue measurable, and Smítal’s construction of it makes use of the continuum hypothesis.

In the literature there is no known chaotic function possessing a scrambled set of positive Lebesgue measure. Here, without using the continuum hypothesis, we construct a continuous chaotic function \( f : [0, 1] \to [0, 1] \), along with an extremally scrambled set \( E \) of Lebesgue measure \( \frac{1}{8} \).

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First we consider the class \( \mathcal{G} \) of all continuous functions \( g: [0, 1] \to [0, 1] \) such that 
\[
g(x) = 3x \quad \text{for} \quad x \in [0, \frac{1}{3}] \quad \text{and} \quad g(x) = 3x - 2 \quad \text{for} \quad x \in \left[ \frac{2}{3}, 1 \right],
\]
and find a set \( K \subset (0, \frac{1}{3}) \) which is an extremally scrambled set of every \( g \in \mathcal{G} \). Any such \( g \) is clearly chaotic. Then we construct a fat Cantor set \( E \subset \left[ \frac{3}{8}, \frac{5}{8} \right] \) with Lebesgue measure \( \frac{1}{8} \). Finally we choose a particular \( f \in \mathcal{G} \) which, when restricted to \( E \), is a monomorphism (actually a homeomorphism) from \( E \) to \( K \). Since \( K \) is an extremally scrambled set of \( f \), \( E \) is one also.

**Preliminaries.** Let \( \mathbb{N} \) denote the natural numbers. Let \( \Omega = \{0, 1\}^{\mathbb{N}} \) be the space of all one-sided sequences of two symbols, along with the dictionary ordering relation \( < \), and the topology of coordinatewise convergence. The shift \( \sigma \) on \( \Omega \) is defined by 
\[
(\sigma \omega)_k = \omega_{k+1}, \quad k \in \mathbb{N}.
\]

Let the map \( \varphi: \Omega \to [0, 1] \) be defined by
\[
\varphi(\omega) = 2 \sum_{k=1}^{\infty} \omega_k 3^{-k} .
\]

Then the image under \( \varphi \) of \( \Omega \) is the usual "middle thirds" Cantor set \( C \), and \( \varphi \) is an order preserving homeomorphism from \( \Omega \) to \( C \). Also, for any \( g \in \mathcal{G} \), we have
\[
g(\varphi(\omega)) = 3\varphi(\omega) - 2\omega_1 = 2 \sum_{k=1}^{\infty} \omega_{k+1} 3^{-k} = \varphi(\sigma \omega).
\]

And, for \( n \in \mathbb{N} \),
\[
g^n(\varphi(\omega)) = \varphi(\sigma^n \omega).
\]

The following simple lemma will prove useful.

**Lemma.** (i) If \( g \in \mathcal{G} \), \( \alpha, \beta \in \Omega \), \( \alpha_{n+j} = \beta_{n+j} \) for \( j = 1, \ldots, k \), then
\[
|g^n(\varphi(\alpha)) - g^n(\varphi(\beta))| \leq 3^{-k} .
\]
(ii) If \( g \in \mathcal{G} \), \( \alpha \in \Omega \), \( \alpha_{n+j} = 0 \) (\( \alpha_{n+j} = 1 \)), for \( j = 1, \ldots, k \), then
\[
g^n(\varphi(\alpha)) \leq 3^{-k}(g^n(\varphi(\alpha)) \geq 1 - 3^{-k}).
\]

**Proof.** The lemma clearly follows from the fact that
\[
g^n(\varphi(\omega)) = \varphi(\sigma^n \omega) = 2 \sum_{j=1}^{\infty} \omega_{n+j} 3^{-j}.
\]

I. We construct a set \( K \subset C \), which is an extremally scrambled set of every \( g \in \mathcal{G} \). The method used here is similar to that used by M. Osikawa and Y. Oono to construct a scrambled set [3].

Let \( r: \mathbb{N} \to \mathbb{N} \) be defined by
\[
r(k) = \inf \left\{ l \in \mathbb{N} \mid k \leq \sum_{j=1}^{l} j^2 + 2j \right\}
\]
and let \( s: \mathbb{N} \to \mathbb{N} \) be defined by
\[
s(k) = \sup \left\{ l \in \mathbb{N} \cup \{0\} \mid l + 1 \leq \left( k - \sum_{j=1}^{r(k)-1} j^2 + 2j \right) / r(k) \right\}
\]
where \( \sum_{j=1}^{0} j^2 + 2j \) is defined to be 0. Then define \( Z: \Omega \to \Omega \) by

\[
(Z(\omega))_k = \begin{cases} 
0 & \text{if } s(k) = 0, \\
1 & \text{if } s(k) = 1, \\
\omega_{s(k)-1} & \text{if } s(k) \geq 2,
\end{cases}
\]

i.e., \( Z(\omega) = 01\omega_10011\omega_1\omega_2\omega_2 \cdots \). Let \( K \) be the set \( \varphi(Z(\Omega)) \).

The map \( Z: \Omega \to Z(\Omega) \) is an order preserving homeomorphism, and since \( \varphi: \Omega \to C \) is also, we see that \( K \) is homeomorphic to \( C \). This fact will be important in §111, where we construct a homeomorphism from the fat Cantor set \( E \) of §11 to \( K \).

**Proposition.** \( K \) is an extremally scrambled set of every \( g \in \mathcal{G} \).

**Proof.** Let \( x, y \in K \), \( x < y \), and \( p \) be any periodic point of \( g \) with period \( q \). It suffices to show that equations (4)-(6) hold with \( g \) in place of \( f \).

Let \( \alpha = \varphi^{-1}(x) \) and \( \beta = \varphi^{-1}(y) \). Then by the construction of \( K \) we have, for any \( k \in \mathbb{N} \), infinitely many \( n \in \mathbb{N} \) such that \( \alpha_{n+j} = \beta_{n+j} = 0 \) for \( j = 1, \ldots, k \). Thus, by (i) of the Lemma, (5) is satisfied. Since \( x < y \), we also have infinitely many \( n \in \mathbb{N} \) such that \( \alpha_{n+j} = 0 \) and \( \beta_{n+j} = 1 \), for \( j = 1, \ldots, k \). Thus, by (ii) of the Lemma, (4) is satisfied. We also have infinitely many \( n \in \mathbb{N} \) such that \( \alpha_{n+j} = 0 \) (\( \alpha_{n+j} = 1 \)) for \( j = 1, \ldots, k + q \) so that

\[
\limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq \max\{g^n(p) | n \in \mathbb{N}\}
\]

\[
\left( \limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq \max\{1 - g^n(p) | n \in \mathbb{N}\} \right)
\]

holds. Therefore,

\[
\limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq 1/2
\]

holds.

II. Here we constructed a fat Cantor set \( E \subset [\frac{3}{8}, \frac{5}{8}] \) so that \( m(E) = \frac{1}{8} \) [4, 5]. The set \( E \) will be the countable intersection of a nested sequence of compact sets \( \{E_n\} \) where each \( E_n \) is the union of \( 2^n \) disjoint closed intervals, exactly two of these intervals being contained in each one of the \( 2^{n-1} \) disjoint closed intervals of \( E_{n-1} \). Also, the diameters of the intervals comprising \( E_n \) go uniformly to 0 as \( n \) goes to \( \infty \).

Let \( \Gamma_n = \{0, 1\}^n \) denote the set of all two symbols of length \( n \), and \( \gamma_k \) is the \( k \)th coordinate of \( \gamma \in \Gamma_n \). There are \( 2^n \) elements of \( \Gamma_n \).

The sine of \( \pi/6 \) can be written as an infinite product [6] as follows,

\[
\sin \frac{\pi}{6} = \frac{\pi}{6} \prod_{l=1}^{\infty} \left( 1 - \frac{1}{36l^2} \right) = \frac{1}{2}.
\]

Let \( E_0 = [\frac{3}{8}, \frac{5}{8}] \) and let \( E_n \) be the union of the \( 2^n \) disjoint closed intervals \( I(\gamma) \), where \( \gamma \in \Gamma_n \). The right-hand endpoint of \( I(\gamma) \) is

\[
b(\gamma) = \frac{5}{8} - \frac{1}{4} \sum_{k=1}^{n} \gamma_k 2^{-k} \frac{\pi}{6} \prod_{l=1}^{6} \left( 1 - \frac{1}{36l^2} \right)
\]
and the left-hand endpoint of $I(\gamma)$ is

$$a(\gamma) = b(\gamma) - \frac{1}{4} 2^{-n} \frac{n+1}{6} \prod_{l=1}^{n+1} \left(1 - \frac{1}{36l^2}\right).$$

It follows that if $n \geq 2$, $\gamma \in \Gamma_n$, $\lambda \in \Gamma_{n-1}$, then either $I(\gamma)$ and $I(\lambda)$ are disjoint, or $I(\gamma)$ is contained in $I(\lambda)$. The latter occurs iff $\gamma_k = \lambda_k$ for $k = 1, \ldots, n - 1$. The Lebesgue measure of $E_n$ is $\frac{1}{4} \frac{5}{6} \prod_{l=1}^{n+1} (1 - 1/36l^2)$ and, since $E_n \upharpoonright E = \bigcap_{n} E_n$, we have $m(E) = \lim_{n \to \infty} E_n = \frac{1}{8}$.

A point is in $E$ iff it is a limit point of the right-hand endpoints of the intervals comprising the sets $E_n$. So, the map $\psi : \Omega \to E$ defined by

$$\psi(\omega) = \frac{5}{8} - \frac{1}{4} \sum_{k=1}^{\infty} \omega_k 2^{-k} \frac{n+1}{6} \prod_{l=1}^{k} \left(1 - \frac{1}{36l^2}\right)$$

is an order reversing homeomorphism.

III. We now choose a particular $f \in \mathcal{G}$ so that $f$ restricted to $E$ is a homeomorphism (monomorphism is all that is necessary) from $E$ to $K$. Then for any $x, y \in E$, $x \neq y$, $p$ any periodic point of $f$, we have $f(x), f(y) \in K$, $f(x) \neq f(y)$, $f(p)$ a periodic point of $f$. Thus $E$ will be an extremally scrambled set of $f$.

Let $f(t) = \varphi(Z(\psi^{-1}(t)))$ for $t \in E$. Since $f \in \mathcal{G}$, we have $f(t) = 3t$ for $t \subset [0, \frac{1}{3}]$ and $f(t) = 3t - 2$ for $t \subset [\frac{2}{3}, 1]$. We now define $f(t)$ for $t \subset E \cap (\frac{1}{3}, \frac{2}{3})$ to be a linear interpolation of the values of $f$ on the nearest points of $E \cup \{\frac{1}{3}\} \cup \{\frac{2}{3}\}$ to $t$.

More precisely, for $t \subset E \cap (\frac{1}{3}, \frac{2}{3})$, we define

$$f(t) = f(t_1) + \frac{(t - t_1)}{(t_r - t_1)} (f(t_r) - f(t_1))$$

where $t_1 = \sup\{t' \in E \mid t' < t\} \cup \{\frac{1}{3}\}$ and $t_r = \inf\{t' \in E \mid t' > t\} \cup \{\frac{2}{3}\}$.

**Proposition.** The function $f$ is continuous and $K$ is the homeomorphic image under $f$ of $E$.

**Proof.** It suffices to show that $f$ is continuous, strictly decreasing on $[\frac{1}{3}, \frac{2}{3}]$, and $f(E) = K$.

The image under $f$ of $E$ is the image under $\varphi \circ Z$ of $\psi^{-1}(E)$, but $\psi^{-1}(E) = \Omega$, so

$$f(E) = \varphi(Z(\psi^{-1}(E))) = \varphi(Z(\Omega)) = K.$$ 

The function $f$ is continuous and strictly decreasing on $E \cup \{\frac{1}{3}\} \cup \{\frac{2}{3}\}$, since $f(\frac{1}{3}) = 1$, $f(\frac{2}{3}) = 0$, $f(E) = K \subset (0, \frac{1}{3})$, and also $\psi^{-1}$ is continuous and order reversing and both $\varphi$ and $Z$ are continuous and order preserving. On the rest of $[\frac{1}{3}, \frac{2}{3}]$, $f$ is just contained by linear interpolation, so $f$ is continuous and strictly decreasing on $[\frac{1}{3}, \frac{2}{3}]$.

**Conclusion.** The $f$ constructed in the third part is a chaotic function possessing an extremally scrambled set $E$ of Lebesgue measure $\frac{1}{8}$.

**Remark.** It is easy to find a chaotic function $h : [0, 1]^r \to [0, 1]^r$ which has a scrambled set of positive $r$ dimensional Lebesgue measure $m_r$. Let $h$ be defined by

$$h(x_1, \ldots, x_r) = (f(x_1), \ldots, f(x_r)).$$
Then $E^r$ is a scrambled set of $h$ and $m_r(E^r) = \left( \frac{1}{3} \right)^r$. In fact, for any $x, y \in E^r$, $x \neq y$, $p$ any periodic point of $h$, we have

\begin{align*}
(7) \quad & \limsup_{n \to \infty} |h^n(x) - h^n(y)| \geq 1, \\
(8) \quad & \liminf_{n \to \infty} |h^n(x) - h^n(y)| = 0, \\
(9) \quad & \limsup_{n \to \infty} |h^n(x) - h^n(p)| \geq \sqrt{r}/2.
\end{align*}

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