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THE NONEXISTENCE OF MAXIMAL INVARIANT MEASURES ON ABELIAN GROUPS

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ABSTRACT. It is shown that every σ-additive σ-finite invariant measure on an abelian group has a proper σ-additive invariant extension.

We consider σ-finite countably additive measures which vanish on points and are nonidentically zero. Throughout this paper the word "measure" will mean a measure enjoying all the above properties. A measure m defined on a σ-algebra S of subsets of X is called invariant with respect to a group G of bijections of X if for any $T \in G$ and $A \in S$ the image $T^*(A)$ is an element of S and $m(T^*(A)) = m(A)$. A measure m defined on a σ-algebra of subsets of a group G is called invariant if it is invariant with respect to left translations.

Sierpiński (quoted in Szpilrajn [**7**]) asked whether there exists in Euclidean n-dimensional space E^n a maximal extension of the Lebesgue measure invariant with respect to the group of isometries of E^n. Hulanicki [**2**] proved that if $|X|$ is less than the first real-valued measurable cardinal, $|G| \leq |X|$, and m is a measure on X invariant with respect to G and vanishing on sets of cardinality $< |X|$, then there exists a proper extension of m invariant with respect to G. Thus he solved Sierpiński's problem under additional set theoretic assumptions. Harazišvili [**1**] gave a negative answer to this question for $n = 1$ without any extra hypotheses. He also proved that there is no maximal measure invariant with respect to translations on any Euclidean space. In other words the group of translations of E^n does not carry maximal invariant measures. Our theorem is a generalisation of the above result.

THEOREM. *Every invariant measure on an abelian group $(G, +)$ has a proper invariant extension.*

PROOF. We start with the following lemma, essentially due to Szpilrajn [**7**]. The easy proof is left to the reader.

LEMMA. *Let m be an invariant measure on G. If there exists a set $E \subset G$ such that:*

1. *E is not a set of m measure zero.*
2. *For every sequence $\{g_n \colon n \in \omega\}$ of elements of G, there exists a sequence $\{h_\alpha \colon \alpha < \omega_1\}$ of elements of G such that for distinct α, β,*

$$m\left(\left[h_\alpha + \bigcup_{n\in\omega}(g_n + E)\right] \cap \left[h_\beta + \bigcup_{n\in\omega}(g_n + E)\right]\right) = 0,$$

then the measure m has a proper invariant extension.

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Hence it suffices to show a set E with the above properties. Without loss of generality we assume that m is a complete measure (i.e. subsets of measure zero sets are measurable).

Case 1. *Additive groups of linear spaces over a countable field* (cf. Harazišvili [**1**] and Pelc [**4**]). Let V be a linear space over a countable field K and m any measure on V invariant with respect to addition. Fix a linear basis $\mathcal{B} = \{V_\alpha\colon \alpha < \kappa\}$ of V over K and let V_n denote the set of those elements of V which have n summands in the basis $\mathcal{B}$ representation.

Hence $V = \bigcup_{n\in\omega} V_n$ and there exists the least number n_0 for which V_{n_0} does not have measure 0. We claim that V_{n_0} also satisfies condition 2 of the lemma.

Let $\{g_n\colon n \in \omega\}$ be a countable sequence of elements of V and

$$D = \bigcup_{n\in\omega} (g_n + V_{n_0}).$$

As $\{h_\alpha\colon \alpha < \omega_1\}$ for the lemma take any subset of $\mathcal{B}$ of cardinality ω_1 whose elements do not appear in the $\mathcal{B}$-representation of $g_i - g_j$ where $(i,j) \in \omega \times \omega$. Then $w = h_\alpha + g_i + w_1 = h_\beta + g_j + w_2$, where w_1 and w_2 are in V_{n_0}, if and only if $g_i - g_j = h_\beta - h_\alpha + w_2 - w_1$. Since h_β and h_α are not used in the $\mathcal{B}$-representation of $g_i - g_j$ and they are distinct, then either $w_1 = kh_\alpha + w'$ or $w_2 = kh_\alpha + w'$ for some $k \in K$ and w' in V_{n_0-1}. Hence $w = k'h_\alpha + g_i + w'$ or $w = h_\beta + g_j + k'h_\alpha + w'$ for some $k' \in K$ and $w' \in V_{n_0-1}$ so that for $\alpha \neq \beta$ the set $(h_\alpha + D) \cap (h_\beta + D)$ is a subset of a countable union of translations of V_{n_0-1}. Therefore $(h_\alpha + D) \cap (h_\beta + D)$ has m measure zero. Hence the set V_{n_0} satisfies the conditions of the lemma.

Case 2. *Torsion-free abelian groups.* Let G be a torsion-free abelian group. There exists a homomorphic embedding of G into the additive group of a linear space V over the field Q of rationals such that a certain basis $\mathcal{B} = \{v_\alpha\colon \alpha < \kappa\}$ of V consists of elements of G. Let m be any invariant measure on G.

For any finite sequence $s = (q_1, \ldots, q_n)$ of nonzero rationals let V_s be the set of elements of V of the form $q_1 v_{\alpha_1} + \cdots + q_n v_{\alpha_n}$ where $\alpha_1 > \cdots > \alpha_n$ and $v_{\alpha_i} \in \mathcal{B}$. Let $s_0 = (r_1, \ldots, r_n)$ be a sequence for which the set $E = G \cap V_{s_0}$ is not a set of m measure 0. In order to check that E also satisfies condition 2 of the lemma, let $\{g_n\colon n \in \omega\}$ be any sequence of elements in G. Take any uncountable set of elements w_α of $\mathcal{B}$ which do not appear in the $\mathcal{B}$-representation of any element g_n. Let k be a natural number different from all r_i, $r_i - r_j$ $(i, j \leq n)$ and $h_\alpha = kw_\alpha$ for $\alpha < \omega_1$. We claim that

$$\left[h_\alpha + \bigcup_{n\in\omega} (g_n + E)\right] \cap \left[h_\beta + \bigcup_{n\in\omega} (g_n + E)\right] = \emptyset .$$

Indeed, suppose x is an element of the set on the left side. Then

$$x = kw_\alpha + g_n + r_1 v_{\alpha_1} + \cdots + r_n v_{\alpha_n} = kw_\beta + g_m + r_1 v_{\beta_1} + \cdots + r_n v_{\beta_n}.$$

Since $\alpha \neq \beta$ and w_α, w_β do not appear in the representation of g_n, g_m, we get that either $k = r_i$ or $k + r_i = r_j$ for some $i, j \leq n$, contradiction.

Case 3. *Arbitrary grpups.* Let G be an arbitrary abelian group and m an invariant measure on G. By H denote the torsion subgroup of G. If $m(H) = 0$ we define a measure m_1 on G/H, putting $m_1(\{a + H\colon a \in A\}) = m(\bigcup_{a\in A}(a + H))$

for $A \subset G$ such that $\bigcup_{a \in A}(a + H)$ is m-measurable. The measure m_1 is clearly invariant (and vanishes on points since $m(H) = 0$). The group G/H is torsion-free and, hence, by Case 2 there exists a set $E_1 \subset G/H$ satisfying both conditions from the lemma for G/H and m_1. It is not hard to see that the set $E = \bigcup E_1$ satisfies the conditions from the lemma for G and m.

If H is not a set of m measure 0 then let H_n (for $n \geq 1$) denote the subgroup of H consisting of those elements whose orders divide n. Clearly $H = \bigcup_{n \geq 1} H$ and let n_0 be the least natural number for which H_{n_0} is not a set of m measure 0. We will prove the existence of a subset of H_{n_0} satisfying the conditions of our lemma by induction on the number of prime divisors of n_0 (counting multiple divisions many times). If $k = 1$ then n_0 is prime and H_{n_0} is the additive group of a linear space over the field F_{n_0}. Next we proceed as in Case 1 and show that the set constructed there is as required (for G and m).

Suppose that for n_0 having k prime divisors there exists a set $E \subset H_{n_0}$ satisfying the lemma. Now let $n_0 = p_1 \cdots p_{k+1}$ (p_i-primes, $k \geq 1$) and let H' be the subgroup of H_{n_0} consisting of elements of order p_1. Since $m(H') = 0$, we can define an invariant measure m' on G/H' just as before. H_{n_0}/H' is a subgroup of G/H' all of whose elements have orders dividing the number $p_2 \cdots p_{k+1}$. By definition H_{n_0}/H' is not a set of m' measure 0. Hence by the inductive hypothesis there exists a set $E' \subset H_{n_0}/H'$ which satisfies the conditions of the lemma for the group G/H' and measure m'. It is easy to see that set $E = \bigcup E'$ is now good for G and m, which finishes the proof in the general case.

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