THE NONEXISTENCE OF MAXIMAL INVARIANT MEASURES ON ABELIAN GROUPS

ANDRZEJ PELC

ABSTRACT. It is shown that every $\sigma$-additive $\sigma$-finite invariant measure on an abelian group has a proper $\sigma$-additive invariant extension.

We consider $\sigma$-finite countably additive measures which vanish on points and are nonidentically zero. Throughout this paper the word "measure" will mean a measure enjoying all the above properties. A measure $m$ defined on a $\sigma$-algebra $S$ of subsets of $X$ is called invariant with respect to a group $G$ of bijections of $X$ if for any $T \in G$ and $A \in S$ the image $T^*(A)$ is an element of $S$ and $m(T^*(A)) = m(A)$. A measure $m$ defined on a $\sigma$-algebra of subsets of a group $G$ is called invariant if it is invariant with respect to left translations.

Sierpiński (quoted in Szpilrajn [7]) asked whether there exists in Euclidean $n$-dimensional space $E^n$ a maximal extension of the Lebesgue measure invariant with respect to the group of isometries of $E^n$. Hulanicki [2] proved that if $|X|$ is less than the first real-valued measurable cardinal, $|G| \leq |X|$, and $m$ is a measure on $X$ invariant with respect to $G$ and vanishing on sets of cardinality $< |X|$, then there exists a proper extension of $m$ invariant with respect to $G$. Thus he solved Sierpiński’s problem under additional set theoretic assumptions. Harazisvili [1] gave a negative answer to this question for $n = 1$ without any extra hypotheses. He also proved that there is no maximal measure invariant with respect to translations on any Euclidean space. In other words the group of translations of $E^n$ does not carry maximal invariant measures. Our theorem is a generalisation of the above result.

THEOREM. Every invariant measure on an abelian group $(G, +)$ has a proper invariant extension.

PROOF. We start with the following lemma, essentially due to Szpilrajn [7]. The easy proof is left to the reader.

LEMMA. Let $m$ be an invariant measure on $G$. If there exists a set $E \subseteq G$ such that:

1. $E$ is not a set of $m$ measure zero.
2. For every sequence $\{g_n: n \in \omega\}$ of elements of $G$, there exists a sequence $\{h_\alpha: \alpha < \omega_1\}$ of elements of $G$ such that for distinct $\alpha, \beta$,

$$m \left( \left( h_\alpha + \bigcup_{n \in \omega} (g_n + E) \right) \cap \left( h_\beta + \bigcup_{n \in \omega} (g_n + E) \right) \right) = 0,$$

then the measure $m$ has a proper invariant extension.
Hence it suffices to show a set $E$ with the above properties. Without loss of generality we assume that $m$ is a complete measure (i.e. subsets of measure zero sets are measurable).

**Case 1.** Additive groups of linear spaces over a countable field (cf. Haraziávili [1] and Pelc [4]). Let $V$ be a linear space over a countable field $K$ and $m$ any measure on $V$ invariant with respect to addition. Fix a linear basis $B = \{V_\alpha: \alpha < \kappa\}$ of $V$ over $K$ and let $V_n$ denote the set of those elements of $V$ which have $n$ summands in the basis $B$ representation.

Hence $V = \bigcup_{n \in \omega} V_n$ and there exists the least number $n_0$ for which $V_{n_0}$ does not have measure 0. We claim that $V_{n_0}$ also satisfies condition 2 of the lemma.

Let $\{g_n: n \in \omega\}$ be a countable sequence of elements of $V$ and

$$D = \bigcup_{n \in \omega} (g_n + V_{n_0}).$$

As $\{h_\alpha: \alpha < \omega_1\}$ for the lemma take any subset of $B$ of cardinality $\omega_1$ whose elements do not appear in the $B$-representation of $g_i - g_j$ where $(i, j) \in \omega \times \omega$. Then $w = h_\alpha + g_i + w_1 = h_\beta + g_j + w_2$, where $w_1$ and $w_2$ are in $V_{n_0}$, if and only if $g_i - g_j = h_\beta - h_\alpha + w_2 - w_1$. Since $h_\beta$ and $h_\alpha$ are not used in the $B$-representation of $g_i - g_j$ and they are distinct, then either $w_1 = kh_\alpha + w'$ or $w_2 = kh_\alpha + w'$ for some $k \in K$ and $w'$ in $V_{n_0 - 1}$. Hence $w = k'h_\alpha + g_i + w'$ or $w = h_\beta + g_j + k'h_\alpha + w'$ for some $k' \in K$ and $w' \in V_{n_0 - 1}$ so that for $\alpha \neq \beta$ the set $(h_\alpha + D) \cap (h_\beta + D)$ is a subset of a countable union of translations of $V_{n_0 - 1}$. Therefore $(h_\alpha + D) \cap (h_\beta + D)$ has measure zero. Hence the set $V_{n_0}$ satisfies the conditions of the lemma.

**Case 2.** Torsion-free abelian groups. Let $G$ be a torsion-free abelian group. There exists a homomorphic embedding of $G$ into the additive group of a linear space $V$ over the field $Q$ of rationals such that a certain basis $B = \{v_\alpha: \alpha < \kappa\}$ of $V$ consists of elements of $G$. Let $m$ be any invariant measure on $G$.

For any finite sequence $s = (q_1, \ldots, q_n)$ of nonzero rationals let $V_s$ be the set of elements of $V$ of the form $q_1 v_{\alpha_1} + \cdots + q_n v_{\alpha_n}$ where $\alpha_1 > \cdots > \alpha_n$ and $v_{\alpha_i} \in B$. Let $s_0 = (r_1, \ldots, r_n)$ be a sequence for which the set $E = G \cap V_{s_0}$ is not a set of $m$ measure 0. In order to check that $E$ also satisfies condition 2 of the lemma, let $\{g_n: n \in \omega\}$ be any sequence of elements in $G$. Take any uncountable set of elements $w_\alpha$ of $B$ which do not appear in the $B$-representation of any element $g_n$. Let $k$ be a natural number different from all $r_i$, $r_i - r_j$ ($i, j \leq n$) and $h_\alpha = kw_\alpha$ for $\alpha < \omega_1$. We claim that

$$\left[ h_\alpha + \bigcup_{n \in \omega} (g_n + E) \right] \cap \left[ h_\beta + \bigcup_{n \in \omega} (g_n + E) \right] = \emptyset.$$

Indeed, suppose $x$ is an element of the set on the left side. Then

$$x = kw_\alpha + g_n + r_1 v_{\alpha_1} + \cdots + r_n v_{\alpha_n} = kw_\beta + g_m + r_1 v_{\beta_1} + \cdots + r_n v_{\beta_n}.$$

Since $\alpha \neq \beta$ and $w_\alpha, w_\beta$ do not appear in the representation of $g_n, g_m$, we get that either $k = r_i$ or $k + r_i = r_j$ for some $i, j \leq n$, contradiction.

**Case 3.** Arbitrary groups. Let $G$ be an arbitrary abelian group and $m$ an invariant measure on $G$. By $H$ denote the torsion subgroup of $G$. If $m(H) = 0$ we define a measure $m_1$ on $G/H$, putting $m_1(\{a + H: \alpha \in A\}) = m(\bigcup_{\alpha \in A}(a + H))$
for $A \subseteq G$ such that $\bigcup_{a \in A} (a + H)$ is $m$-measurable. The measure $m_1$ is clearly invariant (and vanishes on points since $m(H) = 0$). The group $G/H$ is torsion-free and, hence, by Case 2 there exists a set $E_1 \subseteq G/H$ satisfying both conditions from the lemma for $G/H$ and $m_1$. It is not hard to see that the set $E = \bigcup E_1$ satisfies the conditions from the lemma for $G$ and $m$.

If $H$ is not a set of $m$ measure 0 then let $H_n$ (for $n \geq 1$) denote the subgroup of $H$ consisting of those elements whose orders divide $n$. Clearly $H = \bigcup_{n \geq 1} H_n$ and let $n_0$ be the least natural number for which $H_{n_0}$ is not a set of $m$ measure 0. We will prove the existence of a subset of $H_{n_0}$ satisfying the conditions of our lemma by induction on the number of prime divisors of $n_0$ (counting multiple divisors many times). If $k = 1$ then $n_0$ is prime and $H_{n_0}$ is the additive group of a linear space over the field $F_{n_0}$. Next we proceed as in Case 1 and show that the set constructed there is as required (for $G$ and $m$).

Suppose that for $n_0$ having $k$ prime divisors there exists a set $E \subseteq H_{n_0}$ satisfying the lemma. Now let $n_0 = p_1 \cdots p_{k+1}$ ($p_i$-primes, $k \geq 1$) and let $H'$ be the subgroup of $H_{n_0}$ consisting of elements of order $p_1$. Since $m(H') = 0$, we can define an invariant measure $m'$ on $G/H'$ just as before. $H_{n_0}/H'$ is a subgroup of $G/H'$ all of whose elements have orders dividing the number $p_2 \cdots p_{k+1}$. By definition $H_{n_0}/H'$ is not a set of $m'$ measure 0. Hence by the inductive hypothesis there exists a set $E' \subseteq H_{n_0}/H'$ which satisfies the conditions of the lemma for the group $G/H'$ and measure $m'$. It is easy to see that set $E = \bigcup E'$ is now good for $G$ and $m$, which finishes the proof in the general case.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARSAW, PLOW S-NXP, 00-901, WARSAW, POLAND