The family of all functions of the form
\[ f(z) = z + a_2 z^2 + \cdots, \]
regular and one-to-one in the open unit disk \( \Delta, \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \), is (as usual) denoted by \( S \). The class \( S \) has been the subject of much inquiry (see Bernardi [1] for example).

The inverse of \( f(z) \) has a series expansion in a disk centered at the origin with the form
\[ f(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots. \]
The search for bounds on the magnitude of the coefficients \( \gamma_k \) has raised some interesting and unexpected questions (see [4-8] or [3, §7, Chapter 16]).

Let \( \mathcal{P} \) be the family of all functions
\[ P(z) = 1 + c_1 z + c_2 z^2 + \cdots \]
regular in \( \Delta \) and satisfying \( \text{Re}\{P(z)\} > 0 \) for \( z \) in \( \Delta \). The normalized integral of any function in \( \mathcal{P} \) is a member of \( S \) [3], i.e.,
\[ f(z) = \int_0^z P(\zeta) d\zeta \]
is in \( S \) if \( P(z) \) is in \( \mathcal{P} \); we denote all functions formed in this way by \( I \).

Earlier [7] the authors showed that if \( f(z) \) is in \( I \), then the coefficients of \( f(w) \), given in (2), are restricted so that
\[ |\gamma_2| \leq 1, \quad |\gamma_3| \leq \frac{4}{3}, \quad |\gamma_4| \leq \frac{13}{6}, \quad |\gamma_5| \leq \frac{59}{15} \quad \text{and} \quad |\gamma_6| \leq \frac{344}{45}. \]
It was shown that these bounds are rendered sharp only for the inverse of the function
\[ f_0(z) = -z - 2 \log(1 - z), \]
which corresponds, as in (4), to the member \( P_0(z) \) of \( \mathcal{P} \) given by
\[ P_0(z) = \frac{1 + z}{1 - z} = 1 + 2 \sum_{k=1}^{\infty} z^k, \quad \text{for } z \text{ in } \Delta. \]
Furthermore, the authors suggested that it is quite likely that \( f_0(w) \), the inverse of (6), gives sharp upper bounds for other (perhaps all) coefficients exhibited in (2). It is the purpose of this note to show that the authors’ suggestion is correct.

Specifically, the authors propose the following

**Theorem.** If \( f(z) \) is any member of \( I \),

\[
\hat{f}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots
\]

and

\[
f_0(w) = w + B_2 w^2 + B_3 w^3 + \cdots
\]

is the inverse of \( f_0(z) \) given in (6), then

\[
|\gamma_k| \leq |B_k|, \quad k = 2, 3, 4, \ldots,
\]

and \( f_0(w) \) is the unique function giving equality in (8).

**Proof.** If \( P(z) \) is in \( P \) then the reciprocal \( 1/P(z) \) is likewise in \( P \), and conversely. Consequently (4) can be written as \( f'(z) = \sqrt{P(z)} \) and this, combined with the relation \( f(f(w)) = w \), gives the equation

\[
f'(w) = P(f(w)), \quad |w| < \rho(f).
\]

Using representations (2) and (3) enables us to rewrite (9) as

\[
1 + \sum_{k=2}^{\infty} k \gamma_k w^{k-1} = 1 + \sum_{k=1}^{\infty} C_k(\hat{f}(w))^k,
\]

from which, after some work, one obtains

\[
\begin{align*}
2! \cdot \gamma_2 &= C_1, & 3! \cdot \gamma_3 &= C_1^2 + 2C_2, \\
4! \cdot \gamma_4 &= C_1^3 + 8C_1C_2 + 6C_3, \\
5! \cdot \gamma_5 &= C_1^4 + 22C_1^2C_2 + 42C_1C_3 + 16C_2^2 + 24C_4, \\
6! \cdot \gamma_6 &= C_1^5 + 52C_1^3C_2 + 192C_1^2C_3 + 136C_1C_2^2 \\
&\quad + 264C_1C_4 + 180C_2C_3 + 120C_5.
\end{align*}
\]

Furthermore, a careful examination of the formation of the coefficients in (11), and those with higher index, reveals that

\[
k! \cdot \gamma_k = C_1^{k-1} + R_k(C_1, C_2, \ldots, C_{k-1})
\]

for each \( k \), \( R_k \) being a polynomial whose coefficients are all nonnegative.

As a consequence, a sharp upper bound for (12) may be obtained by a direct application of the triangle inequality along with the bounds \(|C_k| \leq 2\) for all \( k \) [3]. The (unique) function extremizing \(|C_1|\) and all subsequent \(|C_k|\) is \( P_0(z) \), given in (7). This concludes the proof.

The function \( \hat{f}_0(w) \) of the theorem has the Maclaurin series

\[
\hat{f}_0(w) = w - w_2 + \frac{4}{3} w^3 - \frac{13}{6} w^4 + \frac{59}{15} w^5 - \frac{344}{45} w^6 + \cdots + B_k w^k + \cdots
\]

and the coefficients are defined recursively by

\[
(k + 1) B_{k+1} + B_k + \sum_{j=1}^{k} (k + 1 - j) B_j B_{k+1-j} = 0,
\]

for \( k = 1, 2, \ldots \) and \( B_1 = 1 \).
In comparison with the authors' earlier attempt [7] of a proof of this theorem, the above is straightforward and simple. This is largely due to the observation that replacing $P(z)$ in the equation $f'(z) = P(z)$ by its reciprocal gives representations for the $\gamma_k$'s (in terms of $C_k$'s as in (11)) in a far more tractable form. This observation was made by Jan Campschroer in a related work [2].

Suppose $f(z)$ is an odd function in $I$. Then its inverse is likewise odd and, in this case, the corresponding function $P(z)$ defined in (4) must be even. An application of the preceding technique justifies the following statement.

**COROLLARY.** If $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ is an odd member of $I$, then the coefficients of its inverse

$$f(w) = w + \beta_3w^3 + \beta_5w^5 + \cdots,$$

satisfy the (sharp) inequalities

$$|\beta_{2k+1}| \leq |C_{2k+1}|, \quad k = 1, 2, \ldots;$$

the numbers $C_{2k+1}$ are the coefficients of the inverse of $g_0(z)$ in $I$, which is obtained by letting $P(z)$ be $P_0(z^2)$ in (4).

The function defined in the corollary is

$$g_0(z) = -z + \log \left( \frac{1+z}{1-z} \right)$$

and its inverse is

$$g_0(w) = w - \frac{2}{3}w^3 + \frac{14}{15}w^5 - \frac{538}{315}w^7 + \frac{13,478}{2,835}w^9 + \cdots.$$  

The functions in $I$ which are $k$-fold symmetric can be handled by the method of the corollary.

[This work was done at Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin, Poland, under an exchange program between the United States National Academy of Sciences and Polska Akademia Nauk.]

**REFERENCES**


**DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711**

**INSTYTUT MATEMATYKI, UMCS, UL. NOWOTKI 10, 20-031 LUBLIN, POLAND**