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ON CERTAIN SUPPORT POINTS OF THE CLASS S

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ABSTRACT. A support point of S maps the unit disk onto the complement of an analytic arc going to ∞. We study the case where this arc has an analytic continuation through its finite endpoint and back to ∞. As an application we find that only under severe restrictions can a Bazilevič function be a support point of S.

Let Δ be the open unit disk in the complex plane, and let $H(\Delta)$ be the usual topological linear space of holomorphic functions on Δ. The class S is the subset of $H(\Delta)$ consisting of univalent functions f with the normalization $f(0) = f'(0) - 1 = 0$. Let L be an element of the dual space $H(\Delta)^*$ of $H(\Delta)$; that is, let L be complex-valued, linear, and continuous on $H(\Delta)$. Then the functional $\operatorname{Re} L$ achieves a maximum on S, and if L is nonconstant on S, any extremal function f maps Δ onto the complement of an analytic arc Γ_f satisfying

$$L\left(\frac{f^2}{f-w}\right)\left(\frac{dw}{w}\right)^2 > 0 \qquad (w \in \Gamma_f) \tag{1}$$

[**6**, **4**, **2**]. (This can fail at the endpoint of Γ_f, but only if f is a rotation of the Koebe function.) Also [**4**, **2**], there is a line to which Γ_f is asymptotic at ∞. Such an extremal function f is called a support point of S.

For the functionals L usually studied—for example coefficient functionals or, more generally, point evaluation of a derivative of some order at a point of Δ—the expression $L(f^2/(f-w))$ is a rational function of w. In this note we assume this is the case and we study the situation where Γ_f has an analytic continuation through its finite endpoint and back to ∞. (Known examples occur when Γ_f is a half line with radial angle at the endpoint less than or equal to $\pi/4$. This is due to K. Pearce [**3**].) We prove that this analytic extension of Γ_f must pass through ∞ analytically, or in other words must be a closed analytic curve on the Riemann sphere. As a consequence we find that no Bazilevič function—except Pearce's examples—can be a support point of S. We have been unable to determine whether the closed analytic curve must be a circle, that is, whether the known examples are the only ones possible. Also unsettled is the question of whether our assumption that $L(f^2/(f-w))$ is rational can be dropped either in the theorem below or in the corollary concerning Bazilevič functions. Our theorem, then, is the following.

THEOREM. *Suppose f is a support point of S with respect to the functional $L \in H(\Delta)^*$. Assume that*

(a) *$L(f^2/(f-w))$ is a rational function of w, and*

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(b) Γ_f, *the omitted arc of* f, *has an analytic extension* Γ *through the finite tip and going back to* ∞.

Then, in a neighborhood of ∞, Γ *is the conformal image of a line segment. In particular there is a (single) line to which* Γ *is asymptotic at both ends.*

PROOF. Let $\Gamma = \gamma(I)$, where I is a real interval and γ is analytic on I. We define the function φ by

$$\varphi(t) = L\left[\frac{f^2}{f-\gamma(t)}\right]\left[\frac{\gamma'(t)}{\gamma(t)}\right]^2, \qquad t \in I. \tag{2}$$

Then by (a), φ is analytic on I except possibly for poles, and by (1), φ is positive on a subinterval of I. It follows that φ is real valued on I (except possibly for poles). Now, the quadratic differential in (1) has a simple pole at ∞ [**7**, **4**], while the "model" for quadratic differentials with a simple pole is $(1/\varsigma)\,d\varsigma^2$ (simple pole at 0) [**5**, p. 213]. This means that the quadratic differential (1) can be obtained from the model by a conformal substitution $\varsigma = h(w)$ taking ∞ to 0, and therefore that the trajectory structure of (1) near ∞ conformally the same as that of the model near 0. (Note that for any analytic arc α near ∞, the value of $L(f^2/(f-w))(dw/w)^2$ at $w = \alpha(t)$ is equal to $(1/\varsigma)\,d\varsigma^2$ at $\varsigma = h(\alpha(t))$.) But the only analytic arcs going to 0 on which $(1/\varsigma)\,d\varsigma^2$ is real lie on $\mathbf{R}$. Hence, near ∞, Γ is the conformal image of an interval $(-\varepsilon, \varepsilon)$ (namely by h^{-1}). The final assertion follows easily. We remark that our proof remains valid as long as $L(f^2/(f-w))$ has only isolated singularities in $\mathbf{C}$.

To apply the theorem to Bazilevič functions we begin by recalling that f is Bazilevič of type $\gamma = \alpha + i\beta$ ($\alpha > 0$, $\beta \in \mathbf{R}$) if

$$f(z) = \left[\int_0^z \gamma g(x)^\alpha h(x) x^{i\beta-1}\,dx\right]^{1/\gamma}. \tag{3}$$

Here g is a normalized ($g(0) = 0$, $g'(0) = 1$) starlike function, and h is a subordinate to some function of the form $(1-az)/(1-z)$, $|a| = 1$. The "half line functions" mentioned before the theorem can of course be obtained by choosing $\gamma = 1$, $g(z) = z/(1-bz)^2$, $h(z) = (1-az)/(1-bz)$ ($|a| = |b| = 1$, $a \neq b$). We suppose now that such a function f is a support point of S. Then $\mathbf{C}\backslash f(\Delta)$ is an analytic arc Γ_f, and it follows from [**1**, Theorem 2] that $\Gamma_f = l^{1/\gamma}$, where l is some half line. (We conjecture that another proof of this fact can be obtained by showing that for arbitrary γ, (3) defines a single-slit mapping if and only if g and h are chosen exactly as above. Then, for $z = e^{i\theta}$,

$$\frac{d}{d\theta}f(z)^\gamma = iz\frac{d}{dz}f(z)^\gamma = i\gamma z^{i\beta}\left[\frac{z}{(1-bz)^2}\right]^\alpha \frac{1-az}{1-bz}.$$

Since $z^{i\beta}$ is positive, $[z/(1-bz)^2]^\alpha$ has only one argument, and $(1-az)/(1-bz)$ has only two possible arguments, it follows that $\{f(e^{i\theta})^\gamma\}$ is a half line.) Calculations show that if $\beta \neq 0$, $l^{1/\gamma}$ has no asymptotic direction at ∞, while if $\beta = 0$ and $0 < \alpha < 1$, $l^{1/\alpha}$ has an asymptotic direction but no asymptotic line. Hence $\gamma = \alpha \geq 1$. Next, assuming f is not a Koebe function, we conclude that the line L containing l does not pass through 0. Therefore Γ_f has the analytic continuation $L^{1/\alpha}$, and condition (b) of the theorem is satisfied. But $L^{1/\alpha}$ satisfies the conclusion of the theorem only if $\alpha = 1$. Indeed, another calculation shows that if $\alpha > 1$, $L^{1/\alpha}$

has two asymptotic rays separated by an angle of π/α. Thus we have proved the following result.

COROLLARY. *Let f be a Bazilevič function and also a support point of S with corresponding function $L(f^2/(f-w))$ rational. Then Γ_f is a half line.*

We conclude by describing a situation in which condition (b) of the theorem obtains (and here we thank Peter Duren for a helpful discussion). Suppose $f \in S$ and f is a support point with respect to two "essentially different" linear functionals, L and J. By this we mean that it is not the case that $L(h) = ah(0) + bh'(0) + cJ(h)$ for certain constants a, b, c and for all $h \in H(\Delta)$. We suppose also that $L(f^2/f-w))$ and $J(f^2/(f-w))$ are rational functions of w with quotient $q(w)$. Then, by (1) and the corresponding version of (1) for J, q is nonnegative on Γ_f. Moreover q is not constant. Indeed,

$$q = c \Rightarrow (L - cJ)(f^2/(f-w)) \equiv 0 \Rightarrow (L - cJ)(h) = ah(0) + bh'(0),$$

contradicting "essentially different". Now from q being rational and nonconstant it can be shown that $q^{-1}(\mathbf{R} \cup \{\infty\})$ is a finite union of closed analytic arcs on the sphere $\mathbf{C} \cup \{\infty\}$. Since Γ_f is a subset of this union, (b) follows. The special case where $L = a_n$ and $J = a_m$ $(m > n \geq 2)$ was discussed by A. K. Bahtin, Ukrainian Math. J., 1981, who showed that f must be a rotation of the Koebe function. It would be interesting to know whether this conclusion holds in the present, more general, context.

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