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THE K-FUNCTIONAL FOR H^1 AND BMO

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ABSTRACT. Peetre's K-functional for the Hardy space H^1 and the space BMO of functions of bounded mean oscillation is explicitly characterized in terms of truncated square functions.

0. Introduction. Given a function for $\mathbf{R}^n$ with $\int_{\mathbf{R}^n}|f(y)|/(1+|y|)^{n+1}\,dy < +\infty$, we let $u(y,t)$ denote its Poisson integral on $\mathbf{R}^{n+1}_+$. We define the truncated Lusin area integral $S_h u$, $0 < h \leqslant +\infty$, by

$$S_h u(x) = \left(\iint_{\Gamma_h(x)} |t\nabla u(y,t)|^2 t^{-n}\, dy \frac{dt}{t}\right)^{1/2},$$

where $\Gamma_h(x)$ denotes the truncated cone

$$\Gamma_h(x) = \left\{(y,t) \in \mathbf{R}^{n+1}_+ : |x-y|_\infty < t/2 < h/2\right\}$$

and $\nabla = (\partial/\partial_{x_1},\ldots,\partial/\partial_{x_n},\partial/\partial t)$. When $h = +\infty$ we drop the subscript h and write Su and Γ, respectively.

The Hardy space $H^1 = H^1(\mathbf{R}^n)$ is the space of all locally integrable functions f on $\mathbf{R}^n$ such that $u(y,t) \to 0$ as $t \to +\infty$, $y \in \mathbf{R}^n$, and

$$\|f\|_{H^1} = \|Su\|_{L^1(\mathbf{R}^n)} < +\infty.$$

There are also many other equivalent ways to define H^1 (cf. **[2, 4, 6]**, for instance).

For a function f on $\mathbf{R}^n$ we also define the sharp maximal function $M^\# f$ by

$$M^\# f(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy$$

where the supremum is taken over all finite cubes Q containing x, and where $f_Q = \int_Q f(y)\,dy/|Q|$. $|Q|$ denotes the Lebesque measure of Q.

The space of functions of bounded mean oscillation BMO = BMO($\mathbf{R}^n$) is the space of all locally integrable functions f on $\mathbf{R}^n$ such that

$$\|f\|_{\mathrm{BMO}} = \|M^\# f\|_{L^\infty(\mathbf{R}^n)} < +\infty.$$

(Strictly speaking, we are going to consider BMO/R which is a Banach space.)

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The purpose of this note is to explicitly characterize the K-functional for the couple (H^1, BMO), i.e. the functional

$$K(t, f; H^1, \mathrm{BMO}) = \inf_{f=f_0+f_1} \left(\|f_0\|_{H^1} + t\|f_1\|_{\mathrm{BMO}}\right)$$

defined for $f \in H^1 + \mathrm{BMO}$ and $t > 0$. This characterization answers a question I was first asked by Professor Colin Bennett at the Williamstown Conference on Harmonic Analysis 1978.

1. The K-functional. We shall characterize the K-functional in terms of the operators $S_\alpha^\#$, $0 < \alpha < 1$, which are defined by (cf. [9])

$$S_\alpha^\# u(x) = \sup_{x\in Q} \inf\left\{A: \left|\left\{y \in Q: S_{h_Q}u(y) > A\right\}\right| < \alpha|Q|\right\},$$

where the supremum is taken over all cubes Q containing x and h_Q is the sidelength of Q. Clearly, these operators satisfy the subadditivity property

$$S_\alpha^\#(u_{f+g})(x) \leqslant 2\left(S_{\alpha/2}^\# u_f(x) + S_{\alpha/2}^\# u_g(x)\right). \tag{1.1}$$

Since $\{S_\alpha^\# u > t\} \subseteqq \{M\chi_{\{Su>t\}} > \alpha\}$, where M is the Hardy-Littlewood maximal operator, it is also immediate that

$$\left\|S_\alpha^\# u\right\|_{L^1} \leqslant c_{\alpha,n}\|f\|_{H^1} \tag{1.2}$$

for $0 < \alpha < 1$. Moreover, Strömberg [9] has shown that

$$\left\|S_\alpha^\# u\right\|_{L^\infty} \leqslant c_{\alpha,n}\|f\|_{\mathrm{BMO}} \tag{1.3}$$

for $0 < \alpha < 1$.

We also recall C. Fefferman's celebrated result according to which $(H^1)^* \approx \mathrm{BMO}$ and BMO can be identified with the collection of linear functionals f determined on $\mathcal{T} = \{\eta: \hat{\eta}$ is C^∞ and has compact support strictly disjoint from the origin$\}$ with

$$\|f\|_* = \sup_{\eta\in\mathcal{T}} |\langle \eta, f\rangle| / \|\eta\|_{H^1} < +\infty.$$

Also, $\|f\|_* \approx \|f\|_{\mathrm{BMO}}$. See [6].

Another fact we shall use is that $\int |f(x)|/(1 + |x|)^{n+1}\, dx < +\infty$ for $f \in H^1 + \mathrm{BMO}$ (cf. [6]).

THEOREM 1.1. *Let $0 < \alpha < \frac{1}{2}$. Then there are constants $c_1 = c_{1;\alpha,n}$ and $c_2 = c_{2;\alpha,n}$ such that*

$$c_1\int_0^t \left(S_\alpha^\# u\right)^*(s)\, ds \leqslant K(t, f; H^1, \mathrm{BMO}) \leqslant c_2\int_0^t \left(S_\alpha^\# u\right)^*(s)\, ds \tag{1.4}$$

for $f \in H^1 + \mathrm{BMO}$ and $t > 0$.

PROOF. That $c_1 \int_0^t (S_\alpha^\# u)^*(s)\, ds \leqslant K(t, f; H^1, \mathrm{BMO})$ is an easy consequence of (1.1–3) and the proof is left to the reader.

Let

$$E(t, f; H^1, \mathrm{BMO}) = \inf_{\|f_1\|_{\mathrm{BMO}} \leqslant t} \|f - f_1\|_{H^1}$$

for $f \in H^1 + \mathrm{BMO}$ and $t > 0$. To prove the converse inequality, it suffices to verify that there is a constant $c = c_{\alpha,n}$ such that

$$E(t, f; H^1, \mathrm{BMO})/t \leqslant c \int_{S_\alpha^\# u > t/c} S_\alpha^\# u(x)\, dx/t. \tag{1.5}$$

Indeed, if this is the case, then (1.4) readily follows by taking the inverse of both sides of (1.5) as functions of t and using that $(E(t, f)/t)^{-1} \approx K(t, f)/t$ and $(\int_{|f|>t} |f(x)|\, dx)/t^{-1} \approx \int_0^t f^*(s)\, ds/t$ (see [**6**]).

Fix $f \in H^1 + \mathrm{BMO}$ and $t > 0$. Essentially following Cohen [**4**] (also cf. [**1, 3**]), we define $\Omega_k = \{y\colon\ S_\alpha^\# u(y) > 2^k t\}$ and $\tilde{\Omega}_k = \{y\colon\ y \in Q$ for some cube Q with $|Q \cap \Omega_k| > \frac{1}{2}|Q|\}$, and let $\{Q_{ik}\}_{i \in \mathbf{Z}}$ be the Whitney decomposition of $\tilde{\Omega}_k$ (see [**8**]), $k \in \mathbf{Z}$. Also, define $E_{ik}^+ = \{(y, t) \in \mathbf{R}_+^{n+1}\colon y \in Q_{ik}, |Q(y, t) \cap \Omega_k| > \frac{1}{2}|Q(y, t)|$ and $|Q(y, t) \cap \Omega_{k+1}| \leqslant \frac{1}{2}|Q(y, t)|\}$, where $Q(y, t)$ is the cube with center y and side-length t.

Select a real-valued, radial C^∞-function Ψ with $\operatorname{supp} \Psi \subseteq \{|y| \leqslant 1\}$ and such that $\hat{\Psi}(0) = 0$ and

$$\int_0^\infty 2\pi r|\xi| e^{-2\pi r|\xi|} \hat{\Psi}(r\xi) \frac{dr}{r} = -1.$$

for any $\xi \neq 0$. Define the linear functionals f_0 and f_1 by

$$\langle \eta, f_0 \rangle = \sum_{k \geqslant 0} \sum_i \iint_{E_{ik}^+} r \frac{\partial u}{\partial r}(y, r)(\Psi_r * \eta)(y)\, dy \frac{dr}{r} = \sum_{k \geqslant 0} \sum_i \langle \eta, a_{ik} \rangle$$

for $\eta \in \mathscr{S}$, and with $E^+ = (\bigcup_{k \geqslant 0} \bigcup_i E_{ik}^+)^c$,

$$\langle \eta, f_1 \rangle = \iint_{E^+} r \frac{\partial u}{\partial r}(y, r)(\Psi_r * \eta)(y)\, dy \frac{dr}{r}$$

for $\eta \in \mathscr{T}$. We claim that $f_0 + f_1$ is the almost optimal decomposition of f we need to prove (1.5). To verify our claim, we shall show that $f = f_0 + f_1$ in $H^1 + \mathrm{BMO}$, that $\|f_0\|_{H^1}/t$ is less than the right-hand side of (1.5) and that $\|f_1\|_{\mathrm{BMO}} \leqslant ct$.

We first estimate the BMO-norm of f_1 by duality.

Let $h(x) = \sup\{h\colon S_h u(x) \leqslant S_\alpha^\# u(x)\}$. Then

$$S_{h(x)} u(x) \leqslant S_\alpha^\# u(x) \tag{1.6}$$

and

$$\left|\{x \in Q\colon h(x) < h_Q\}\right| \leqslant \alpha |Q| \tag{1.7}$$

for each cube Q. Pick $\eta \in \mathscr{T}$ with $\|\eta\|_{H^1} = 1$. We then have

$$|\langle \eta, f_1 \rangle| \leqslant c \iint_{E^+} \left(\left|\{z \in \Omega_0^c\colon (y, r) \in \Gamma_{h(z)}(z)\}\right| / r^n \right) \\ \times \left| r \frac{\partial u}{\partial r}(y, r) \right| |\Psi_r * \eta(y)| dy \frac{dr}{r}$$

because of the definition of the E_{ik}'s and (1.7) (here we are using $\alpha < 1/2$). By Fubini's theorem, Schwarz's inequality and (1.6), this is less than

$$c\int_{\Omega_0^c} S_{h(z)}u_f(z)S_{h(z),\Psi}\eta(z)\,dz \leqslant c\int_{\Omega_0^c} S_\alpha^\# u_f(z)S_\Psi\eta(z)\,dz$$
$$\leqslant ct\|S_\Psi\eta\|_{L^1}$$

where

$$S_{h,\Psi}\eta(z) = \left(\iint_{\Gamma_h(z)} |\Psi_r * \eta(y)|^2\,dy\,\frac{dr}{r}\right)^{1/2}$$

and $S_\Psi\eta(z) = S_{\infty,\Psi}\eta(z)$. It is well known (cf. [**2**, **4**]) that $\|S_\Psi\eta\|_{L^1} \leqslant c\|\eta\|_{H^1}$ and we thus find

$$|\langle \eta, f_1\rangle| \leqslant ct\|\eta\|_{H^1}.$$

According to the above-mentioned result of C. Fefferman, this implies $\|f_1\|_{\mathrm{BMO}} \leqslant ct$.

To estimate f_0, we observe that if $(y,t) \in E_{ik}^+$ then $Q(y,t) \subseteqq 10\sqrt{n}\,Q_{ik}$ since $Q(y,t) \subseteqq \tilde{\Omega}_k$ and $\operatorname{dist}(y, \tilde{\Omega}_k^c) \leqslant 5\operatorname{diam} Q_{ik}$. If we thus replace E^+ by E_{ik}^+, Ω_0^c by $\Omega_{k+1}^c \cap 10\sqrt{n}\,Q_{ik}$ and $\|\eta\|_{H^1} = 1$ by $\|\eta\|_{L^2} = 1$ in the above, it follows that $\|a_{ik}\|_{L^2} \leqslant c2^k t|10\sqrt{n}\,Q_{ik}|^{1/2}$. It is also easy to see that $\operatorname{supp} a_{ik} \subseteqq 10\sqrt{n}\,Q_{ik}$ and $\int a_{ik}(x)\,dx = 0$. Consequently, $a_{ik}/(c2^k t|Q_{ik}|)$ is a (1,2)-atom for an appropriate constant c and

$$\sum_{k\geqslant 0}\sum_i \|a_{ik}\|_{H^1} \leqslant c\sum_{k\geqslant 0}\sum_i 2^k t|Q_{ik}| \leqslant c\sum_k 2^k t|\tilde{\Omega}_k|.$$

By the Hardy-Littlewood maximal theorem, $|\tilde{\Omega}_k| \leqslant c|\Omega_k|$ and, hence, $f_0 \in \mathscr{S}'$ with

$$\|f_0\|_{H^1} \leqslant c\sum_{k\geqslant 0} 2^k t|\Omega_k| \leqslant c\int_{S_\alpha^\# u > t/c} S_\alpha^\# u(x)\,dx.$$

There only remains to verify that $f = f_0 + f_1$ in $H^1 + \mathrm{BMO}$. Whenever $f \in L^2$, it follows by taking Fourier transforms that

$$\langle \eta, f\rangle = \iint r\frac{\partial u}{\partial r}(y,r)\Psi_r * \eta(y)\,dy\,\frac{dr}{r}, \qquad \eta \in \mathscr{T}.$$

By a routine argument this identity extends to general $f \in H^1 + \mathrm{BMO}$ (approximate f by $f_N(x) = f(x)$ if $|x| \leqslant N$, 0 otherwise and use that $\Psi_r * \eta(y)$ is rapidly decreasing in y and t). Hence, the tempered distributions f and $f_0 + f_1$ satisfy $\langle \eta, f\rangle = \langle \eta, f_0 + f_1\rangle$ for $\eta \in \mathscr{T}$, which implies that $f(x) = f_0(x) + f_1(x) +$ polynomial. However, $\int |f(x)|/(1 + |x|)^{n+1}\,dx < +\infty$ and similarly for $f_0 + f_1$, and the polynomial must be a constant. This proves $f = f_0 + f_1$ in $H^1 + \mathrm{BMO}$, since we are working modulo constants in $H^1 + \mathrm{BMO}$, and completes the proof of the theorem.

REMARK 1.2. The theorem states that $K(t, f; H^1, \mathrm{BMO}) \approx K(t, S_\alpha^\# u; L^1, L^\infty)$, $0 < \alpha < 1/2$. This should be compared with the result of C. Fefferman, N. Rivière and Y. Sagher [**5**] that $K(t, f; H^1, L^\infty) \approx K(t, \Phi^* f; L^1, L^\infty)$ where $\Phi^* f$ is the "grand maximal function".

REMARK 1.3. A similar, but simpler, proof shows that $K(t, f; H^{p_0}, H^{p_1}) \approx K(t, Su; L^{p_0}, L^{p_1})$, $0 < p_0, p_1 < +\infty$.

REMARK 1.4. It is possible to show that the inequality (1.3) can be reversed (see [9], Theorem 5.3) so that $\|f\|_{\mathrm{BMO}} \approx \|S_\alpha^\# u\|_{L^\infty}$. The same is also true for (1.2) when $0 < \alpha < 1/2$: Suppose $f \in \mathscr{S}'$ with $\|S_\alpha^\# u\|_{L^1} < +\infty$. Then, arguing as in the proof above, we find that

$$\sum_k \sum_i c2^k|Q_{ik}|\left(a_{ik}/c2^k|Q_{ik}|\right)$$

is an atomic decomposition of f with

$$\sum_k \sum_i c2^k|Q_{ik}| \leqslant c\|S_\alpha^\# u\|_{L^1}.$$

Hence, $\|f\|_{H^1} \approx \|S_\alpha^\# u\|_{L^1}$, $0 < \alpha < 1/2$.

ADDED IN PROOF. (1) The restriction $0 < \alpha < 1/2$ in Theorem 1.1, Remark 1.2 and Remark 1.4 is easily removed. (2) Essentially the same proof as above also shows that $K(t, f; H^p, \mathrm{BMO}) \approx K(t, S_\alpha^\# u; L^p, L^\infty)$, $0 < \alpha < 1$, for $0 < p \leqslant 1$. (3) Professor R. deVore has independently obtained another characterization of the K-functional for H^1 and BMO. His characterization is in terms of H^1-versions of the sharp maximal function.

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