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ON A PROBLEM OF HELLERSTEIN, SHEN AND WILLIAMSON

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ABSTRACT. Suppose that f is a nonentire transcendental meromorphic function, real on the real axis, such that f and f' have only real zeros and poles, and f' omits a nonzero value. Confirming a conjecture of Hellerstein, Shen and Williamson, it is shown that then f is essentially $f(z) = \tan z - Bz - C$ for suitable values of B and C.

The purpose of this note is to prove the following theorem.

THEOREM 1. *Suppose that f is a nonentire, transcendental meromorphic function, real on the real axis with only real poles, and that the zeros of f and f' are real. If f' omits a nonzero value, then the omitted value is real and*

$$f(z) = A[\tan(az + b) - B(az + b) - C], \tag{1}$$

where A, B, C, a, b, are real, $A \neq 0 \neq a$, $B \geqslant 1$, and C has the following property. Let n be an integer such that

$$-\tfrac{1}{2}B\pi < C_0 = C + nB\pi \leqslant \tfrac{1}{2}B\pi.$$

Then with $\beta = (\sqrt{B-1} + |C_0|)/B$, we have $\beta < \frac{1}{2}\pi$ and $\tan^2\beta \leqslant B - 1$. Further, the zeros of f'' are real.

Theorem 1 is connected to the problem of determining all meromorphic functions f with only real poles such that f, f' and f'' have only real zeros. By a result of Hellerstein, Shen and Williamson [**4**, Theorems 1 and 2], the only remaining open case is when f is real (i.e. $f(z)$ is real or ∞ if z is real) and not entire. For such transcendental functions f the only known examples are given by (1) and

$$f(z) = A\tan(az + b) + B, \tag{2}$$

where A, b, a, b are real and $A \neq 0 \neq a$. For f given by (1) (by (2)), f' omits a nonzero value (omits zero). Hellerstein, Shen and Williamson announced [**5**] that if f is a real, transcendental, nonentire solution to the problem and f' omits zero, then f is given by (2). Theorem 1 confirms their conjecture that if the value omitted by f' is nonzero instead, then f is given by (1). We note that they only had (1) with $B = 1$, $C = 0$. Here we do not need the assumption that the zeros of f'' are real.

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To prove Theorem 1, we first note that if f' omits α, $\alpha \neq 0$, then the equations $f = 0$, $f = \infty$ and $f' = \alpha$ have only real solutions. Hence by a theorem of Edrei [**2**] the order of f does not exceed one. We can write

$$f'(z) = \alpha + 1/g(z), \tag{3}$$

where g is a transcendental entire function of order at most one with only real zeros and with at least one zero. Each zero of g has order two at least.

We show that α is real. If not, then $f' = h/k$, where h and k are real entire functions of order one at most, with no common zeros and only real zeros. By (3), $h - \alpha k = k/g$ is entire, not real, nonvanishing and of order one at most. Thus $(h - \alpha k)(z) = Ae^{Bz}$ and $(h' - \alpha k')(z) = ABe^{Bz}$, $A \neq 0$, where not both A and B are real. Since g has only multiple zeros, so does k, and so we can find a real X such that $k(X) = k'(X) = 0$. Since h is real, so are Ae^{BX} and ABe^{BX}. This means both B and A are real, which is a contradiction. And so α and hence g are real.

Next we show that all the zeros of g have order two. If not, let X_0 be a (real) zero of g of order $k \geqslant 3$. Since f' has only real zeros and its order does not exceed one, the function

$$g(z) + \alpha^{-1} = \alpha^{-1} + A_k(z - X_0)^k + \cdots \tag{4}$$

has only real zeros and thus belongs to the Laguerre-Polya class U_0 (cf. [**3** pp. 227, 228]). By a result of Polya and Schur [**6**, pp. 104, 121], such a function cannot have two or more successive vanishing coefficients between two nonvanishing ones in its Taylor series about a real point. This contradicts (4), so that all zeros of g have order two. Hence all poles of f are simple, and $g = h^2$ for some real entire function h of order one at most, belonging to the Laguerre-Polya class.

The residue of f', i.e. the residue of $1/h^2$, must vanish at every zero X_0 of h. This implies that $h''(X_0) = 0$. Hence $h''/h = k$, say, is entire, since the zeros of h are simple. The lemma of the logarithmic derivative and the fact that h has finite order imply that k is a polynomial. Since the order of h is at most one, well-known results [**7**, pp. 68–70] (also cf. [**1**, Theorem 1]) imply that k is a constant. Since h is transcendental, real and in U_0 with at least one zero, it follows that $h(z) = A_1\cos(A_2 z + A_3)$ for some real A_1, A_2, A_3 with $A_1 \neq 0 \neq A_2$.

Now integration of (3) gives

$$f(z) = A[\tan(az + b) - B(az + b) - C]$$

for real A, B, C, a, b with $A \neq 0 \neq a$, $B \neq 0$. The zeros of f'' are then real, whereas $f^{(3)}$ has nonreal zeros. The zeros of f' are real if and only if $B \geqslant 1$, and f' omits $\alpha = -aAB$. (To see this note that $\tan z$ is real if and only if z is real, and $\tan^2 z \neq -1$ for all z.)

To complete the proof of Theorem 1, it remains to show that the zeros of

$$f(z) = \tan z - Bz - C, \qquad B \geqslant 1, \tag{5}$$

are real if and only if C is admissible, i.e. has the property mentioned in Theorem 1. Since

$$f(z) = \tan(z - n\pi) - B(z - n\pi) - C_0,$$

we may assume that $C = C_0$. Then since $-B\pi/2 < C \leqslant B\pi/2$ the reader may verify that f has

(i) exactly one zero in I_n, $n = \pm 1, \pm 2, \ldots$,

(ii) exactly three (one) zeros in I_0,

if C is admissible (not admissible), where

$$I_n = \left[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right].$$

Now

$$f(z) = \frac{-G'(z)}{\cos z}\exp\left[-\frac{Bz^2}{2} - Cz\right], \tag{6}$$

where

$$G(z) = \cos z \exp(Bz^2/2 + Cz).$$

By Rolle's theorem, G' has at least one zero in each of the I_n. But since $B > 0$, $G \in U_2$ (for the definition see e.g. [3]) and so by Lemma 8 in [3], G' has exactly two additional zeros in all of $\mathbf{C}$. This, together with (i), (ii) and (6), shows that f has only real zeros if and only if C is admissible. The proof of Theorem 1 is now complete.

We would like to thank Hellerstein and Williamson for bringing Lemma 8 in [3] to our attention. This considerably shortened our original proof.

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