ON A PROBLEM OF HELLERSTEIN, SHEN AND WILLIAMSON

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Abstract. Suppose that \( f \) is a nonentire transcendental meromorphic function, real on the real axis, such that \( f \) and \( f' \) have only real zeros and poles, and \( f' \) omits a nonzero value. Confirming a conjecture of Hellerstein, Shen and Williamson, it is shown that then \( f \) is essentially \( f(z) = \tan z - Bz - C \) for suitable values of \( B \) and \( C \).

The purpose of this note is to prove the following theorem.

Theorem 1. Suppose that \( f \) is a nonentire, transcendental meromorphic function, real on the real axis with only real poles, and that the zeros of \( f \) and \( f' \) are real. If \( f' \) omits a nonzero value, then the omitted value is real and

\[
f(z) = A[\tan(az + b) - B(az + b) - C],
\]

where \( A, B, C, a, b \) are real, \( A \neq 0 \neq a, B \geq 1 \), and \( C \) has the following property. Let \( n \) be an integer such that

\[
-\frac{1}{2}B\pi < C_0 = C + nB\pi \leq \frac{1}{2}B\pi.
\]

Then with \( \beta = (\sqrt{B - 1} + |C_0|)/B \), we have \( \beta < \frac{1}{2}\pi \) and \( \tan^2 \beta \leq B - 1 \). Further, the zeros of \( f'' \) are real.

Theorem 1 is connected to the problem of determining all meromorphic functions \( f \) with only real poles such that \( f, f' \) and \( f'' \) have only real zeros. By a result of Hellerstein, Shen and Williamson [4, Theorems 1 and 2], the only remaining open case is when \( f \) is real (i.e. \( f(z) \) is real or \( \infty \) if \( z \) is real) and not entire. For such transcendental functions \( f \) the only known examples are given by (1) and

\[
f(z) = A\tan(az + b) + B,
\]

where \( A, b, a, b \) are real and \( A \neq 0 \neq a \). For \( f \) given by (1) (by (2)), \( f' \) omits a nonzero value (omits zero). Hellerstein, Shen and Williamson announced [5] that if \( f \) is a real, transcendental, nonentire solution to the problem and \( f' \) omits zero, then \( f \) is given by (2). Theorem 1 confirms their conjecture that if the value omitted by \( f' \) is nonzero instead, then \( f \) is given by (1). We note that they only had (1) with \( B = 1 \), \( C = 0 \). Here we do not need the assumption that the zeros of \( f'' \) are real.

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To prove Theorem 1, we first note that if $f'$ omits $\alpha, \alpha \neq 0$, then the equations $f = 0, f = \infty$ and $f' = \alpha$ have only real solutions. Hence by a theorem of Edrei [2] the order of $f$ does not exceed one. We can write

$$f'(z) = \alpha + 1/g(z),$$

where $g$ is a transcendental entire function of order at most one with only real zeros and with at least one zero. Each zero of $g$ has order two at least.

We show that $\alpha$ is real. If not, then $f' = h/k$, where $h$ and $k$ are real entire functions of order one at most, with no common zeros and only real zeros. By (3), $h - ak = k/g$ is entire, not real, nonvanishing and of order one at most. Thus $(h - ak)(z) = Ae^{Bz}$ and $(h' - ak')(z) = ABBe^{Bz}, A \neq 0$, where both $A$ and $B$ are real. Since $g$ has only multiple zeros, so does $k$, and so we can find a real $X$ such that $k'(X) = 0$. Since $h$ is real, so are $Ae^{Bx}$ and $ABBe^{Bx}$. This means both $B$ and $A$ are real, which is a contradiction. And so $\alpha$ and hence $g$ are real.

Next we show that all the zeros of $g$ have order two. If not, let $X_0$ be a (real) zero of $g$ of order $k \geq 3$. Since $f'$ has only real zeros and its order does not exceed one, the function

$$g(z) + \alpha^{-1} = \alpha^{-1} + Ak(z - X_0)^k + \cdots$$

has only real zeros and thus belongs to the Laguerre-Polya class $U_0$ (cf. [3 pp. 227, 228]). By a result of Polya and Schur [6, pp. 104, 121], such a function cannot have two or more successive vanishing coefficients between two nonvanishing ones in its Taylor series about a real point. This contradicts (4), so that all zeros of $g$ have order two. Hence all poles of $f$ are simple, and $g = h^2$ for some real entire function $h$ of order one at most, belonging to the Laguerre-Polya class.

The residue of $f'$, i.e. the residue of $1/h^2$, must vanish at every zero $X_0$ of $h$. This implies that $h''(X_0) = 0$. Hence $h''/h = k$, say, is entire, since the zeros of $h$ are simple. The lemma of the logarithmic derivative and the fact that $h$ has finite order imply that $k$ is a polynomial. Since the order of $h$ is at most one, well-known results [7, pp. 68–70] (also cf. [1, Theorem 1]) imply that $k$ is a constant. Since $h$ is transcendental, real and in $U_0$ with at least one zero, it follows that $h(z) = A_1 \cos(A_2z + A_3)$ for some real $A_1, A_2, A_3$ with $A_1 \neq 0 \neq A_2$.

Now integration of (3) gives

$$f(z) = A[\tan(az + b) - B(az + b) - C],$$

for real $A, B, C, a, b$ with $A \neq 0 \neq a, B \neq 0$. The zeros of $f''$ are then real, whereas $f'(3)$ has nonreal zeros. The zeros of $f'$ are real if and only if $B \geq 1$, and $f'$ omits $\alpha = -aAB$. (To see this note that $\tan z$ is real if and only if $z$ is real, and $\tan^2 z \neq -1$ for all $z$.)

To complete the proof of Theorem 1, it remains to show that the zeros of

$$f(z) = \tan z - Bz - C, \quad B \geq 1,$$

are real if and only if $C$ is admissible, i.e. has the property mentioned in Theorem 1. Since

$$f(z) = \tan(z - n\pi) - B(z - n\pi) - C_0,$$
we may assume that $C = C_0$. Then since $-B\pi/2 < C \leq B\pi/2$ the reader may verify that $f$ has

(i) exactly one zero in $I_n$, $n = \pm 1, \pm 2, \ldots$,
(ii) exactly three (one) zeros in $I_0$,

if $C$ is admissible (not admissible), where

$$I_n = \left[ (2n - 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{2} \right].$$

Now

$$f(z) = \frac{-G'(z)}{\cos z} \exp \left[ -\frac{Bz^2}{2} - Cz \right],$$

where

$$G(z) = \cos z \exp (Bz^2/2 + Cz).$$

By Rolle's theorem, $G'$ has at least one zero in each of the $I_n$. But since $B > 0$, $G \in U_2$ (for the definition see e.g. [3]) and so by Lemma 8 in [3], $G'$ has exactly two additional zeros in all of $\mathbb{C}$. This, together with (i), (ii) and (6), shows that $f$ has only real zeros if and only if $C$ is admissible. The proof of Theorem 1 is now complete.

We would like to thank Hellerstein and Williamson for bringing Lemma 8 in [3] to our attention. This considerably shortened our original proof.

REFERENCES


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