FUNCTIONS OF WELL-BOUNDED OPERATORS

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ABSTRACT. It is shown that if $A$ is a well-bounded operator of type (B), and if $f$ is of bounded variation and piecewise monotone, then $f(A)$ is also well bounded of type (B).

1. Introduction. Let $C$ be a simple, nonclosed rectifiable arc in the complex plane. For a complex valued function $f$ defined on $C$, let

$$\|f\|_C = \sup(|f|, C) + \text{var}(f, C),$$

where $\sup(|f|, C)$ is the supremum on $C$ of $|f|$, and $\text{var}(f, C)$ is the total variation of $f$ on $C$. Let $X$ denote a complex Banach space and let $A$ denote a bounded linear operator on $X$.

1.1. Definition. The operator $A$ is well bounded if there exist an arc $C$ and a constant $K > 0$ such that for all polynomials $p$, $\|p(A)\| \leq K \|p\|_C$.

Well-bounded operators are important in the spectral theory of linear operators with spectral expansions which are only conditionally convergent, e.g. Fourier expansions in $L^p$ spaces, $p \neq 2$. See [1] for applications to groups and semigroups of linear operators, and to ordinary differential operators. §2 of [1] contains a summary of the theory of well-bounded operators. A systematic development is found in [2, Part 5]. The value of the theory arises from the existence of a functional calculus: a homomorphism from a Banach algebra of functions on $C$ into the algebra of bounded linear operators on $X$. Furthermore, there is a modified Riemann-Stieltjes integral on $C$ which gives a representation of the homomorphism. See below, and also [1, Propositions 2.1, 2.3]. The functional calculus always exists on the algebra of absolutely continuous functions on $C$, but for the special class of well-bounded operators of type (B) (see below, and also [2, p. 315]), and thus always in the case that $X$ is reflexive, the functional calculus can be extended to $BV(C)$, the algebra of functions of bounded variation on $C$, with norm $\|f\|_C$.

In this paper we show that for well-bounded operators of type (B), there is a substantial subset of functions $f$ in $BV(C)$ such that $f(A)$ is also well bounded.

Since the arc $C$ is the image of a finite, closed real interval under the arc-length parameterization, there is no loss of generality in assuming that $C$ is in fact a real interval (see [1, Proposition 2.8]).

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1 Part of this work was done while the author was on sabbatical leave at the University of Duisburg and the Technical University of Aachen, West Germany.
1.2. Definition. A spectral family on the interval \( J = [a, b] \) is a projection-valued function \( E: J \to \mathcal{B}(X) \) (the space of bounded linear operators on \( X \)) satisfying:

(i) \( \sup\{\|E(\lambda)\| : \lambda \in J\} = K < \infty \);
(ii) \( E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min(\lambda, \mu)) \) for \( \lambda, \mu \in J \);
(iii) \( E(\lambda) \) is right continuous on \( J \) in the strong operator topology;
(iv) \( E(\lambda) \) has a left-hand limit in the strong operator topology at each point of \( J \);
(v) \( E(b) = I \) (the identity operator on \( X \)).

For \( f \in \text{BV}(J) \), let \( \int_{J} f(\lambda) \, dE(\lambda) \) denote the strong limit of Riemann-Stieltjes sums such that the intermediate point in each interval is the right endpoint. For the proof that this limit exists, see [2, Chapter 17]. Let

\[
\int_{J} f(\lambda) \, dE(\lambda) = f(a)E(a) + \int_{J} f(\lambda) \, dE(\lambda).
\]

1.4. Theorem [2, Chapter 17]. The bounded linear operator \( A \) is a well-bounded operator of type (B) on \( J \) if and only if there exists a spectral family \( E(\lambda) \) on \( J \) such that

\[
A = \int_{J} \lambda \, dE(\lambda).
\]

For every \( f \in \text{BV}(J) \),

\[
f(A) = \int_{J} f(\lambda) \, dE(\lambda)
\]
defines an algebra homomorphism of \( \text{BV}(J) \) into \( \mathcal{B}(X) \), such that

\[
\|f(A)\| \leq K\|f\|_J.
\]

1.8. Definition. A real-valued function \( f \) on \( J \) is piecewise monotone if \( J \) is the union of finitely many intervals \( J_i = [a_{i-1}, a_i], i = 1, \ldots, m \), with \( a_0 = a \), \( a_m = b \), such that \( f \) is monotone on each \( J_i \).

We shall prove the following theorem.

1.9. Theorem. If \( f \in \text{BV}(J) \) is piecewise monotone, and if \( A \) is a well-bounded operator of type (B) on \( J \), then \( f(A) \) is a well-bounded operator of type (B) (on any compact interval containing \( f(J) \)).

1.10. Remark. For the case \( f \) is strictly monotone and continuous, see [1, Lemma 4.3].

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2. The case that \( f \) is monotone. In this section we prove Theorem 1.9 in the case that \( f \) is monotone on all of \( J \). Let \( A \) be a well-bounded operator of type (B) on \( J \) with spectral family \( \{E(\lambda) : \lambda \in J\} \). Assume \( f \) is bounded and monotone nondecreasing on \( J \), with \( \alpha = f(a), \beta = f(b), H = [\alpha, \beta] \). For \( \mu \) in \( H \), define

\[
S(\mu) = \{\lambda \in J : F(\lambda) \leq \mu\}, \quad \lambda(\mu) = \sup\{\lambda : \lambda \in S(\mu)\}.
\]
Note that
\begin{equation}
S(\mu) = \begin{cases} 
[a, \lambda(\mu)] & \text{if } f \text{ is left continuous at } \lambda(\mu), \\
[a, \lambda(\mu)] & \text{otherwise}.
\end{cases}
\end{equation}

We define a family of projections $F(\mu)$ on $H$ by
\begin{equation}
F(\mu) = \begin{cases} 
E(\lambda(\mu)) & \text{if } f \text{ is left continuous at } \lambda(\mu), \\
E(\lambda'(\mu)) & \text{otherwise}.
\end{cases}
\end{equation}

$(E(\lambda_0) x = \lim_{\lambda \uparrow \lambda_0} E(\lambda) x$, which exists for all $x \in X$ and all $\lambda_0 \in J$ by property (iv) of a spectral family.)

Let $\{\nu_k : k = 1, 2, \ldots \}$ denote the at most countable set in $J$ where $f$ fails to be left continuous, and define
\begin{equation}
\gamma_k = \lim_{\lambda \uparrow \nu_k} f(\lambda), \quad \rho_k = f(\nu_k) - \gamma_k > 0,
\end{equation}
\begin{equation}
h(\lambda) = \begin{cases} 
0, & \lambda \neq \nu_k, \\
\rho_k, & \lambda = \nu_k
\end{cases}
\end{equation}

Then $g$ is left continuous on $J$. (If $\nu_k$ is a cluster point of the set of left discontinuities, we use the fact that $\sum \rho_k < \infty$.)

2.6. Lemma. \{ $F(\mu) : \mu \in H$ \} is a spectral family on $H$.

Proof. $F(\mu)$ clearly satisfies properties (i), (ii), (v) of Definition 1.2. Property (iv) holds for $F$ since $\lambda(\mu)$ is a monotone nondecreasing function of $\mu$, and since (iv) holds for $E$. To establish the right continuity of $F$, let $\mu_0 \in H$, $x \in X$. Let $\lambda_0 = \lim_{\mu \downarrow \mu_0} \lambda(\mu)$. Then $\lambda_0 = \lambda(\mu_0)$, since otherwise there exists $\lambda, \lambda(\mu_0) < \lambda < \lambda_0$, such that $f(\lambda) > \mu_0$ but $f(\lambda) \leq \mu$ for all $\mu > \mu_0$. Suppose first that $f$ is left continuous at $\lambda_0 = \lambda(\mu_0)$. Then for $\mu > \mu_0$,
\begin{equation}
F(\mu) x - F(\mu_0) x = \begin{cases} 
E(\lambda(\mu)) x - E(\lambda_0) x & \text{if } f \text{ is left continuous at } \lambda(\mu), \\
E(\lambda'(\mu)) x - E(\lambda_0) x & \text{otherwise}.
\end{cases}
\end{equation}

In each case, the right continuity of $E(\lambda)$ guarantees that this vector can be made arbitrarily small by selecting $\mu > \mu_0$ sufficiently close to $\mu_0$. Now consider the case that $f$ is not left continuous at $\lambda_0$. Then $f(\lambda_0) > \lim_{\lambda \uparrow \lambda_0} f(\lambda) = \gamma_0$, so for $\gamma_0 \leq \mu < f(\lambda_0)$, $F(\mu)$ is constant and therefore right continuous.

Let $B$ denote the well-bounded operator of type (B) on $H$ given by
\begin{equation}
B = \int_H^{\oplus} \mu \, dF(\mu).
\end{equation}

We prove that $f(A)$ is well bounded by showing

2.8. Lemma. $f(A) = B$.

Proof. First we note that since $\sum \rho_k < \infty$ and $\|E(\lambda)\| \leq K$, we have
\begin{equation}
h(A) = \sum_{k=1}^{\infty} \rho_k [E(\nu_k) - E(\nu_k^-)]
\end{equation}
in the strong operator topology. Let \( x \in X \) and \( \varepsilon > 0 \) be given. Let \( \{ \lambda_k \}_{0}^{n} \) be a partition of \( J \) containing the points \( r_1, \ldots, r_m \) for some \( m \), such that
\[
\left\| g(A)x - \sum_{1}^{n} g(\lambda_k) \left[ E(\lambda_k) - E(\lambda_{k-1}) \right] x \right\| < \varepsilon,
\]
\[
\left\| h(A)x - \sum_{1}^{m} \rho_k \left[ E(\nu_k) - E(\nu_{k-1}) \right] x \right\| < \varepsilon.
\]
Let \( \{ \mu_j \}_{0}^{q} \) denote the partition of \( H \) consisting of all points \( f(\lambda_k) \) where \( \lambda_k \) is in the above partition, and include also those points \( \gamma_i \) corresponding to \( f(\nu_i) \). With respect to this partition,
\[
\sum_{1}^{q} \mu_j \left[ F(\mu_j) - F(\mu_{j-1}) \right]
\]
\[
= \sum_{1}^{m} f(\nu_k) \left[ E(\nu_k) - E(\nu_{k-1}) \right] + \sum_{1}^{n} g(\lambda_k) \left[ E(\lambda_k) - E(\lambda_{k-1}) \right]
\]
\[
= \sum_{1}^{m} \rho_k \left[ E(\nu_k) - E(\nu_{k-1}) \right] + \sum_{1}^{q} \mu_j \left[ E(\lambda_k) - E(\lambda_{k-1}) \right].
\]
Refining the \( \{ \mu_j \} \) partition, if necessary, we have
\[
\| Bx - \sum_{j=1}^{q} \mu_j \left[ F(\mu_j) - F(\mu_{j-1}) \right] x \| < \varepsilon.
\]
Since this only induces a further refinement of the partition of \( J \), we have \( \| Bx - f(A)x \| < 2\varepsilon. \)

2.10. Lemma. If \( f: J \to \mathbb{R} \) is bounded and monotone nonincreasing, then \( f(A) \) is well bounded of type (B).

Proof. It suffices to show that \( -A \) is well bounded of type (B) if \( A \) is. Directly from the definition of total variation, we see that \( \| p(-A) \| \leq K \| p \|_{\infty} \), so \( -A \) is well bounded. To show that \( -A \) is of type (B), it suffices to show that for every \( x \) in \( X \), \( f \to f(-A)x \) is a compact linear map of \( AC(J) \) into \( X \) [2, Theorem 17.14, (ii)]. Since this property holds for \( A \), and \( -A = g(A) \), with \( g(\lambda) = -\lambda \), the result follows.

3. The case that \( f \) is piecewise monotone. Let \( E(\lambda) \) be a spectral family on \( J \), and assume \( J = \bigcup_{1}^{m} J_i \), where \( J_i = [a_{i-1}, a_i], a_0 = a, a_m = b \). We define subspaces \( X_i \) of \( X, i = 0, \ldots, m \):
\[
X_0 = \{ E(a)x : x \in X \},
\]
\[
X_i = \{ [E(\lambda) - E(a_{i-1})] x : x \in X, \lambda \in J_i \}, \quad i = 1, \ldots, m.
\]
Using the defining properties of a spectral family, it is easy to see that each \( X_i \) is closed, and any two have only the zero of \( X \) in common. Since any \( x \in X \) has the decomposition
\[
x = \sum_{i=0}^{m} \left[ E(a_i)x + \sum_{i=1}^{m} [E(a_i) - E(a_{i-1})] x, \right.
\]
we have
\[
X = X_0 \oplus X_1 \oplus \cdots \oplus X_m.
\]
In each $X_i$ there is a spectral family $\{E_i(\lambda) : \lambda \in J_i\}$ given by
\begin{align*}
E_0(\lambda) &= E(a), & \lambda \in J_0 = \{a\}, \\
E_i(\lambda) &= E(\lambda) - E(a_{i-1}), & \lambda \in J_i.
\end{align*}
Let
\begin{equation}
A = \int_{J_0}^a \lambda \, dE(\lambda), \quad A_i = \int_{J_i} \lambda \, dE_i(\lambda).
\end{equation}
By considering a partition of $J$ including the endpoints $a_i$, we have
\begin{equation}
A = A_0 \oplus \cdots \oplus A_m,
\end{equation}
and more generally, if $f \in BV(J), f_i = f|_{J_i}$, we have
\begin{equation}
f(A) = f_0(A_0) \oplus \cdots \oplus f_m(A_m).
\end{equation}
If $f$ is monotone on each $J_i$, then by the considerations of §2, each $B_i = f_i(A_i)$ is well bounded of type (B). To show that $B = f(A)$ is well bounded, let $p$ be a polynomial on $J$. Then
\begin{align*}
\|p(B)\| &\leq \|p_0(B_0)\| + \cdots + \|p_m(B_m)\| \\
&\leq K_0\|p_0\|_{J_0} + \cdots + K_m\|p_m\|_{J_m} \\
&\leq K\left[|p(a)| + \sum_{i}^m \max(\|p_i\|_{J_i}) + \sum_{i}^m \text{var}(p, J_i)\right] \\
&\leq K\|p\|_{J}.
\end{align*}
To show that $B$ is of type (B), we again consider the compactness of the mapping of $AC(J) \to X$ given by $f \to f(B)x, x$ fixed. Since the map $f_i \to f_i(B_i)x_i$ is compact, we easily see that the same holds for $B$.

The spectral family $\{F(\mu) : \mu \in H\}$ of $f(A)$ can be expressed in terms of $E(\lambda)$. For given $\mu \in H$, the level set $S(\mu)$ now consists of the union of finitely many disjoint intervals, which may or may not contain some of their endpoints. Consider the following list giving a correspondence between intervals and differences of projections:
\begin{align*}
(a, \beta), & \quad E(\beta) - E(a), \\
(a, [\beta), & \quad E(\beta) - E(a), \\
[a, \beta], & \quad E(\beta') - E(a), \\
[a, \beta), & \quad E(\beta) - E(\alpha').
\end{align*}
Then $F(\mu)$ is the sum over each interval in $S(\mu)$, of the corresponding differences of projections.

We now indicate the procedure for replacing $J$ by a simple, nonclosed rectifiable arc $C$. Let $\rho_c$ denote the arc-length parameterizations of $C$. Thus there is a real interval $J_c$ whose length is the arc-length of $C$, such that $\rho_c: J_c \to C, \rho_c \in AC(J_c)$, and $|\rho_c'| = 1$ a.e. [3, p. 634]. Let $f \in BV(C)$, and assume $f(C)$ is contained in a simple, nonclosed rectifiable arc $\Gamma$, with arc-length parameterization $\rho_\Gamma: J_\Gamma \to \Gamma$. Define $g: J_c \to J_\Gamma$ by
\begin{equation}
g = \rho_\Gamma^{-1} \circ f \circ \rho_c.
\end{equation}

3.8. Definition. $f$ is piecewise monotone on $C$ if $g$ is piecewise monotone on $J_c$.  

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3.9. **Theorem.** If $A$ is well bounded of type (B) on $C$, and if $f \in \text{BV}(C)$ maps $C$ into $\Gamma$, such that $f$ is piecewise monotone on $C$, then $f(A)$ is well bounded of type (B) on $\Gamma$.

4. **An application.** Let $L$ be a closed operator with domain $D(L)$ dense in $X$. Assume zero is in the resolvent set of $L$, and let $R = (-L)^{-1}$.

4.1. **Theorem.** Assume $R$ is a well-bounded operator of type (B) on a simple, nonclosed rectifiable arc $C$. Let $p$ be a polynomial such that

(i) $p(C)$ is contained in a simple, nonclosed rectifiable arc $\Gamma$;

(ii) $p(-\lambda^{-1}) \neq 0$ for $\lambda$ in $C - \{0\}$.

Then $p(L)$ has a well-bounded inverse of type (B).

**Proof.** Let $R = \int_C^{\oplus} \lambda \, dE(\lambda)$. If $f \in \text{BV}(C)$ and $\alpha > 0$, let

$$f_\alpha(\lambda) = \begin{cases} f(\lambda), & \rho_c(\lambda, 0) \geq \alpha, \\ 0, & \rho_c(\lambda, 0) < \alpha. \end{cases}$$

Then [1, Corollary 5.13]

$$Lx = \lim_{\alpha \downarrow 0} \int_C^{\oplus} \left( \frac{-1}{\lambda} \right) \alpha \, dE(\lambda) x, \quad x \in D(L),$$

and for any polynomial $p$,

$$p(L)x = \lim_{\alpha \downarrow 0} \int_C^{\oplus} p(\lambda) \left( \frac{-1}{\lambda} \right) \alpha \, dE(\lambda) x, \quad x \in D(p(L)).$$

Let $q(\lambda) = \lambda^n p(-1/\lambda)$, where $n$ is the degree of $p$. Then $q$ is also a polynomial of degree $n$, $q(\lambda) \neq 0$ for $\lambda$ in $C - \{0\}$, and $-p^{-1}(-\lambda^{-1}) = -\lambda^n q^{-1}(\lambda)$. By (ii), $-\lambda^n q^{-1}(\lambda) \in \text{BV}(C)$, so

$$S = \int_C^{\oplus} -\lambda^n q^{-1}(\lambda) \, dE(\lambda)$$

is a bounded linear operator. Using the functional calculus,

$$-p(L)Sx = \lim_{\alpha \downarrow 0} \int_C^{\oplus} (1) \alpha \, dE(\lambda) x = x, \quad x \in X,$$

and

$$-Sp(L)x = x, \quad x \in D(p(L)).$$

Thus $S$ is the inverse of $-p(L)$, and $S$ is well bounded of type (B) since the rational function $-\lambda^n q^{-1}(\lambda)$ is piecewise monotone.

**References**


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