PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 1, September 1984

A SHORT PROOF OF A DECOMPOSITION THEOREM OF A VON NEUMANN ALGEBRA

M. AWAMI AND A. B. THAHEEM

ABSTRACT. Let M be a von Neumann algebra and S and T be commuting *-automorphisms on M satisfying the equation: $S + S^{-1} = T + T^{-1}$. It is proved that M can be decomposed by a central projection p in M such that $S = T$ on Mp and $S = T^{-1}$ on $M(1-p)$.

Recently the second author proved the following decomposition theorem.

THEOREM 1 [3, THEOREM 2.6]. *Let S and T be commuting *-automorphisms of a von Neumann algebra M satisfying the operator equation*

$$S + S^{-1} = T + T^{-1}.$$

Then M can be decomposed by a central projection p in M such that $S = T$ on Mp and $S = T^{-1}$ on $M(1-p)$.

Some applications of this result are discussed by Haagerup and Skau in the geometrical interpretation of the Tomita-Takesaki theory (cf. [1]). A noncommutative version of Theorem 1 (in the case of automorphism groups) has been studied in [4] with its proof depending on Arveson's theory of spectral subspaces (cf. [5]). The proof of Theorem 1 is rather long and relies on several technical lemmas. The purpose of this note is to give a short proof of this theorem. In fact, we prove a stronger form of this result (Theorem 2) and get the proof of Theorem 1 as an immediate corollary.

THEOREM 2. *Let S and T be *-automorphisms of a von Neumann algebra M satisfying the operator equation*

$$\text{(i)} \qquad S + T^{-1}S^{-1}T = T + T^{-1}.$$

Then M can be decomposed by a central projection p in M such that $S = T$ on Mp and $S = T^{-1}$ on $M(1-p)$.

*Received by the editors December 14, 1982.
1980 *Mathematics Subject Classification*. Primary 46L10; Secondary 47C15.
Key words and phrases. Automorphisms, central projection.

PROOF. Consider the *-automorphism $S^{-1}T$ on M. By [**2**, Theorem 2] there is a central projection p in M such that $M(1-p)$ is the smallest subalgebra generated by $R(S^{-1}T-1)$, the range of $(S^{-1}T-1)$, and $(S^{-1}T-1)(Mp)=0$. Since

$$(T^{-1}-S)(S^{-1}T-1)=T^{-1}S^{-1}T-T-T^{-1}+S=0 \quad (\text{by (i)}),$$

therefore, $(T^{-1}-S)(M(1-p))=0$. This completes the proof of the theorem.

COROLLARY. *If S and T commute then we get the proof of Theorem* 1.

REFERENCES

1. U. Haagerup and F. Skau, *Geometric aspects of the Tomita-Takesaki theory*. II, Math. Scand. **48** (1981), 241–252.

2. A. B. Thaheem, *Decomposition of a von Neumann algebra relative to a *-automorphism*, Proc. Edinburgh Math. Soc. (2) **22** (1979), 9–10.

3. _____, *Decomposition of a von Neumann algebra*, Rend. Sem. Mat. Univ. Padova **65** (1981), 1–7.

4. A. B. Thaheem, A. Van Daele and L. Vanheeswijck, *A result on two-one-parameter groups of automorphisms*, Math. Scand. (to appear).

5. A. Van Daele, *Arveson's theory of spectral subspaces*, Nieuw Arch. Wisk. (3) **27** (1979), 215–237.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, GARYOUNIS UNIVERSITY, P. O. BOX 9480, BENGHAZI, LIBYA