

## THE CONSISTENCY STRENGTH OF CERTAIN STATIONARY SUBSETS OF $\mathcal{P}_\kappa\lambda$

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ABSTRACT. If  $\kappa \leq \lambda$  are uncountable cardinals with  $\kappa$  regular, let  $S(\kappa, \lambda) = \{x \in \mathcal{P}_\kappa\lambda : |x \cap \kappa| < |x|\}$ . We investigate the consistency strength of the statement " $S(\kappa, \lambda)$  is stationary in  $\mathcal{P}_\kappa\lambda$ ," and prove that it is strictly weaker than " $\exists$  a Ramsey cardinal," which combines with the lower bound ( $0^\#$ ) proven earlier by J. Baumgartner to give a narrow range of the consistency strength of this statement. In addition, we give an example ( $L[U]$ ) to show that " $\exists \lambda > \kappa(S(\kappa, \lambda)$  is stationary)" does not necessarily imply " $S(\kappa, \kappa^+)$  is stationary."

Let  $\kappa$  be an uncountable regular cardinal and let  $\lambda \geq \kappa$ . Consider the following subset of  $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$ :

$$S(\kappa, \lambda) = \{x \in \mathcal{P}_\kappa\lambda : |x \cap \kappa| < |x|\}.$$

In the process of investigating generalizations of the Mahlo hierarchy of cardinals (see [1 and 2]), the following natural question quickly arose:

When is  $S(\kappa, \lambda)$  a stationary subset of  $\mathcal{P}_\kappa\lambda$ ?

For different reasons, James Baumgartner was led to consider the same set, and he provided a partial answer to the above question in the following unpublished result:

1. THEOREM (BAUMGARTNER). *If  $S(\kappa, \lambda)$  is stationary in  $\mathcal{P}_\kappa\lambda$  for some  $\kappa$  and  $\lambda$ , then  $0^\#$  exists.*

Thus, large cardinal considerations come into view. Establishing an upper bound on the consistency strength is an easy exercise for anyone familiar with supercompact cardinals:

2. THEOREM. *If  $\kappa$  is  $\kappa^+$ -supercompact, then  $S(\kappa, \kappa^+)$  is stationary in  $\mathcal{P}_\kappa\kappa^+$ .*

The following theorem provides a couple of trivial restrictions on  $\kappa$  and  $\lambda$ . We leave the easy proof to the reader.

3. THEOREM. *If  $S(\kappa, \lambda)$  is stationary in  $\mathcal{P}_\kappa\lambda$ , then  $\kappa$  is weakly inaccessible and  $|\lambda| > \kappa$ .*

In comparing Theorems 1 and 2 we note that the gap in consistency strength between  $0^\#$  and a  $\kappa^+$ -supercompact cardinal is very large. The main result of this paper is to show that the desired consistency strength lies at the lower end of this range and that one of the conclusions of Theorem 3 cannot be improved. In particular, if  $\kappa$  is weakly inaccessible and  $\lambda > \kappa$  is Ramsey, then  $S(\kappa, \lambda)$  is stationary in  $\mathcal{P}_\kappa\lambda$ .

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4. THEOREM. *Let  $\kappa$  be weakly inaccessible and suppose  $R$  is a stationary subset of  $\kappa$ . Suppose  $\lambda \rightarrow ([\kappa^{<\kappa}]^+)_2^{\leq\omega}$  and let  $\rho < \kappa$ . Then*

$$S = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in R \text{ and } |x| \geq \aleph_\rho(|x \cap \kappa|)\}$$

*is stationary in  $\mathcal{P}_\kappa\lambda$  (where  $\aleph_\rho(|x \cap \kappa|)$  is the  $\rho$ th cardinal successor of  $|x \cap \kappa|$ ).*

PROOF. Let  $C$  be a closed-unbounded subset of  $\mathcal{P}_\kappa\lambda$ . We show that  $S \cap C \neq \emptyset$ . Consider the structure

$$\mathcal{A} = (V_\lambda, \in, C, (\alpha)_{\alpha < \kappa}, \kappa, (h_\alpha)_{\alpha < \kappa}),$$

where  $C$  is a unary predicate,  $(\alpha)_{\alpha < \kappa}$  and  $\kappa$  are constants, and  $(h_\alpha)_{\alpha < \kappa}$  provide a complete set of Skolem functions for  $\mathcal{A}$  such that for each cardinal  $\beta < \kappa$ ,  $(h_\alpha)_{\alpha < \beta}$  is a complete set of Skolem functions for the model

$$\mathcal{A}_\beta = (V_\lambda, \in, C, (\alpha)_{\alpha < \beta}, \kappa, (h_\alpha)_{\alpha < \beta}).$$

Since  $\lambda \rightarrow ([\kappa^{<\kappa}]^+)_2^{\leq\omega}$ , let  $I \subseteq \lambda$  be a set of indiscernibles for  $\mathcal{A}$  of order type  $[\kappa^{<\kappa}]^+$ . Let  $\beta < \kappa$  be a cardinal and let  $\mathcal{B}_\beta$  be the substructure of  $\mathcal{A}_\beta$  generated by  $I$ . Let  $t$  be any Skolem term and let  $y \subseteq \kappa$ ,  $|y| < \kappa$ . Then standard indiscernibility arguments give that if  $t(\bar{a}) = y$  for some increasing sequence  $\bar{a}$  from  $I$ , then  $t(\bar{b}) = y$  for every increasing sequence from  $I$ , since there are only  $\kappa^{<\kappa}$  such  $y$ 's and  $[\kappa^{<\kappa}]^+$  elements of  $I$ . Since there are only  $\beta$  such terms  $t$ , we get that  $|\{x \cap \kappa : x \in \mathcal{B}_\beta \cap \mathcal{P}_\kappa\lambda\}| = \beta$ . Thus there is some  $\gamma < \kappa$  such that  $x \cap \kappa \subseteq \gamma$  for every  $x \in \mathcal{B}_\beta \cap \mathcal{P}_\kappa\lambda$ . Let  $f(\beta)$  be the least such  $\gamma$ . The function  $f$  is clearly a continuous function, so there is a cardinal  $\beta \in R$  such that  $f(\beta) = \beta$ . Now let  $\mathcal{D}$  be the substructure of  $\mathcal{B}_\beta$  generated by the first  $\aleph_\rho(\beta)$  elements of  $I$ . Let  $D = C \cap \mathcal{D}$ . Then  $D$  is a directed subset of  $C$  of cardinality less than  $\kappa$ , so  $x = \bigcup D \in C$ . But it is easy to see that  $x \cap \kappa = \beta$  and  $|x| \geq \aleph_\rho(\beta)$  (since it contains all of the indiscernibles used to generate  $D$ ).

The above theorem shows that relatively small (above  $0^\#$ ) large cardinal properties for  $\lambda$  are enough to get a positive answer to our question. However, when we consider what large cardinal properties for  $\kappa$  will make  $S(\kappa, \lambda)$  stationary, it seems that Theorem 2 is about the best that can be done. As shown in the next theorem,  $\kappa$  measurable is not enough, since  $L[U] \models "0^\dagger \text{ does not exist}"$ .

5. THEOREM. *If  $\kappa$  is measurable,  $\lambda > \kappa$ , and  $S(\kappa, \lambda)$  is stationary in  $\mathcal{P}_\kappa\lambda$ , then  $0^\dagger$  exists.*

PROOF. This is essentially Baumgartner's proof of Theorem 1 with some minor adjustments. Let  $U$  be a normal ultrafilter over  $\kappa$  and consider the structure  $(L_\lambda[U], \in, U)$ . The set

$$C = \{X \in \mathcal{P}_\kappa\lambda : (X, \in, U \cap X) \prec (L_\lambda[U], \in, U)\}$$

is a closed-unbounded subset of  $\mathcal{P}_\kappa\lambda$ , so there is an  $X \in C$  with  $|x \cap \kappa| < |x|$ . If  $L_\beta[D]$  is the transitive collapse of  $X$  and  $\alpha = \overline{X \cap \kappa}$ , then  $|\alpha| < |\beta|$ . Therefore all subsets of  $\alpha$  which appear in  $L[D]$  have already appeared in  $L_\beta[D]$ , so  $L[D] = D$  is a normal ultrafilter over  $\alpha$ . Thus  $0^\dagger$  exists (see [7]).

Finally, we wish to consider the special case  $\lambda = \kappa^+$ . Here the arguments of Theorem 4 are of no use and the following result shows that the statements " $\exists \lambda > \kappa (S(\kappa, \lambda) \text{ is stationary in } \mathcal{P}_\kappa\lambda)$ " and " $S(\kappa, \kappa^+) \text{ is stationary in } \mathcal{P}_\kappa\kappa^+$ " are not equivalent.

6. THEOREM. Let  $U$  be a normal ultrafilter over  $\rho$  and assume  $V = L[U]$ . Then for every  $\kappa$ ,  $S(\kappa, \kappa^+)$  is not stationary in  $\mathcal{P}_\kappa \kappa^+$ .

PROOF. Suppose  $S(\kappa, \kappa^+)$  is stationary in  $\mathcal{P}_\kappa \kappa^+$ . By Theorem 5, clearly  $\kappa \neq \rho$ . If  $\kappa < \rho$ , let  $N = L_{\rho^+}[U]$  and let  $M$  be the transitive collapse of the Skolem hull of  $\kappa^+ \cup \{\kappa^+\} \cup \{f\}$ , where  $f: \kappa^+ \rightarrow \mathcal{P}(\kappa)$  is from the canonical well-ordering of  $\mathcal{P}(\kappa)$  in  $L[U]$ . Let  $C = \{x \in \mathcal{P}_\kappa(M) : f \in x \prec M\}$  and pick  $x \in C$  such that  $|x \cap \kappa| < |x \cap \kappa^+|$ . If  $M_0$  is the transitive collapse of  $x$ , then let  $j: M_0 \rightarrow M$  be the elementary embedding thus induced. Now  $M_0 = L_\beta[D]$  for some  $D$ , and if  $\alpha$  is the critical point of  $j$ , then  $(j^{-1}f)$  must agree with the canonical ordering of subsets of  $\alpha$  in  $L[U]$  (by a usual iterated ultrapower argument which "compares"  $L_\beta[D]$  and  $L[U]$ ). Thus  $L_\beta[D]$  has all subsets of  $\alpha$  and the ultrafilter  $D_0$  defined by  $X \in D_0$  iff  $\alpha \in jX$  is an  $\alpha$ -complete ultrafilter over  $\alpha$ , contradicting that  $\rho$  is the only measurable cardinal in  $L[U]$ . The case  $\kappa > \rho$  is similar.

7. THEOREM. Let  $K$  be the core model (see [4]), and suppose  $\kappa^+ = [\kappa^+]^K$ . Then  $S(\kappa, \kappa^+)$  is not stationary in  $\mathcal{P}_\kappa \kappa^+$ .

PROOF. Suppose  $S(\kappa, \kappa^+)$  is stationary and use the same argument as Theorem 6 (use a mouse for  $M$ ) to get an inner model with  $\alpha < \kappa$  measurable. But this clearly implies that  $(\kappa^+)^K < \kappa^+$ , since iterated ultrapowers can be used to get an inner model in which some  $\lambda > \kappa$  with  $|\lambda| = \kappa$  is measurable. The same argument works on many of Mitchell's more general core models (see [9]).

Theorem 7 tells us that existence of a Ramsey cardinal is not enough (at least in terms of direct implication) to get  $S(\kappa, \kappa^+)$  stationary for some  $\kappa$ , since Mitchell has proven [8] that if  $\kappa$  is Ramsey, then  $\kappa$  is Ramsey in  $K$ .

In attempting to get a nice lower bound on the consistency of " $S(\kappa, \kappa^+)$  is stationary in  $\mathcal{P}_\kappa \kappa^+$ ", the following question seems to be about the easiest one that is not immediately ruled out by one of the theorems in this paper:

If  $\kappa$  is weakly inaccessible,  $\lambda > \kappa$  is Ramsey, and the Lévy collapse is used to change  $\lambda$  to  $\kappa^+$ , does  $S(\kappa, \lambda)$  remain stationary in the generic extension?

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