CHARACTERIZING THE TOPOLOGY OF INFINITE-DIMENSIONAL $\sigma$-COMPACT MANIFOLDS

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ABSTRACT. A metric space $(X, d)$, which is a countable union of finite-dimensional compacta, is a manifold modelled on the space $l_2^f = \{(x_i) \in l_2: \text{all but finitely many } x_i = 0\}$ iff $X$ is an ANR and the following condition holds: given $\varepsilon > 0$, a pair of finite-dimensional compacta $(A, B)$ and a map $f: A \to X$ such that $f|B$ is an embedding, there is an embedding $g: A \to X$ such that $g|B = f|B$ and $d(f(x), g(x)) < \varepsilon$ for all $x \in A$. An analogous condition characterizes manifolds modelled on the space $\Sigma = \{(x_i) \in l_2: \sum_{i=1}^{\infty} (iz_i)^2 < \infty\}$.

1. Introduction. In this note we will deal with the manifolds modelled on the following pre-Hilbert spaces:

$$l_2^f = \{(x_i) \in l_2: \text{all but finitely many } x_i = 0\}$$

and

$$\Sigma = \left\{(x_i) \in l_2: \sum_{i=1}^{\infty} (iz_i)^2 < \infty\right\}.$$

The spaces $l_2^f$ and $\Sigma$ represent the minimal and maximal topological types of infinite-dimensional, $\sigma$-compact, locally convex metric linear spaces in the following sense: every infinite-dimensional, $\sigma$-compact, locally compact metric linear space contains a topological copy of $l_2^f$ and can be topologically embedded in $\Sigma$ (see [4, p. 274]). Several natural pairs of infinite-dimensional spaces have a structure of $(l_2^f, l_\Sigma)$-manifolds (cf. [3, 5, 8, 10]). To recognize them the following characterization was elaborated (cf. [1, 3, 14, 16, 20]): the pair $(M, N)$ of metric spaces is an $(l_2^f, l_\Sigma)$-manifold pair iff $M$ is an $l_2^f$-manifold, $N$ is the countable union of finite-dimensional compacta and the following condition holds:

(1) given $\varepsilon > 0$, a pair $(A, B)$ of finite-dimensional compacta and a map $f: (A, B) \to (M, N)$ such that $f|B$ is an embedding, there exists an embedding $v: A \to N$ such that $v|B = f|B$ and $d(f(x), v(x)) < \varepsilon$ for all $x \in A$, where $d$ is a metric on $M$.

This condition can be used to recognize $l_2^f$-manifolds. But there are situations when we do not know if a given space has a completion homeomorphic to $l_2$ and therefore (1) cannot be applied (e.g. in the case of an $\aleph_0$-dimensional, nonlocally convex, metric linear space). In [9, TC] a question is posed for intrinsic topological characterizations of $l_2^f$-manifolds and $\Sigma$-manifolds without considering suitable completions. In this note we give the following characterizations:

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Let \((X,d)\) be an absolute neighborhood retract which is a countable union of finite-dimensional compacta. Then \(X\) is an \(l_f^d\)-manifold iff the following condition holds:

(2) given a pair \((A,B)\) of finite-dimensional compacta, a map \(f:A \to X\) such that \(f|B\) is an embedding, and \(\varepsilon > 0\), there is an embedding \(v:A \to X\) such that \(v = f\) on \(B\) and \(d(f(x), v(x)) < \varepsilon\) for \(x \in A\).

If a \(\sigma\)-compact ANR space \(Y\) satisfies the condition (2) for every pair of compacta, then it is a \(\Sigma\)-manifold.

These results are obtained, analogously as in \([18\text{ and } 19]\), by considering the projections \(p_X: X \times l_f^d \to X\) and \(p_Y: Y \times \Sigma \to Y\), respectively, and are based on a theorem of Toruńczyk \([17]\) stating that \(X \times l_f^d\) is an \(l_f^d\)-manifold for \(\sigma\)-finite-dimensional \(\sigma\)-compact ANR space \(X\), and \(Y \times \Sigma\) is a \(\Sigma\)-manifold for \(\sigma\)-compact ANR space \(Y\). Since the spaces under consideration are incomplete we cannot use Bing’s shrinking criterion applied in \([18, 19]\). Instead of Bing’s shrinking criterion we use some lemma concerning a stabilizing sequence of maps (see \([13, \text{Lemma 4]}\) and §2 here).

To unify the proofs for the \(l_f^d\)- and \(\Sigma\)-case an abstract scheme of approximation of maps by homeomorphisms is given in §3. The characterizations of \(l_f^d\)- and \(\Sigma\)-manifolds are formulated in §4.

Applying our characterization, it is not hard to prove that every \(\aleph_0\)-dimensional linear metric space (i.e. having a Hamel basis of cardinality \(\aleph_0\)) is homeomorphic to \(l_f^d\) This fact and other consequences of our criteria are given in \([6]\).

2. Preliminaries. In this section we fix notation and formulate some facts needed later.

Suppose that \(X\) and \(Y\) are topological spaces. We write \(\text{cov}(X)\) for family of all open covers of \(X\) and \(C(X,Y)\) for the space of all continuous functions from \(X\) to \(Y\) topologized by the “limitation topology” in which each \(f \in C(X,Y)\) has the collection \(\{V(f, U): U \in \text{cov}(Y)\}\) as a basis of neighborhoods, where \(V(f, U) = \{g \in C(X,Y): \text{for each } x \in X \text{ there exists } U \in U \text{ containing both } f(x) \text{ and } g(x)\}\). Members of \(V(f, U)\) are said to be \(U\)-close to \(f\).

Suppose that \(Y\) is a metric space and \(d\) is a metric on \(Y\). For \(\alpha \in C(Y,(0,1))\) and \(f \in C(X,Y)\), let

\[V(f, \alpha) = \{g \in C(X,Y): d(f(x), g(x)) < \alpha(f(x)) \text{ for each } x \in X\} .\]

The members of \(V(f, \alpha)\) are said to be \(\alpha\)-close to \(f\). A map \(h: X \times [0,1] \to Y\) is said to be an \(\alpha\)-homotopy if, for each \(x \in X\), \(\text{diam}(h(\{x\} \times [0,1])) < \alpha(h(x,0))\).

The following facts are known (cf. Theorem 4.1 of \([4]\) and \([13]\), \([15]\)):

(A) For every \(U \in \text{cov}(Y)\) there exists \(\alpha \in C(Y,(0,1))\) such that for every \(f \in C(X,Y)\) \(V(f, U) \supset V(f, \alpha)\).

(B) For every \(U \in \text{cov}(Y)\) there exists a metric \(\rho\) on \(Y\) compatible with \(d\) such that the cover of \(Y\) by open balls of radius 1 (with respect to \(\rho\)) is a refinement of \(U\).

The map \(f \in C(X,Y)\) is a \textit{near-homeomorphism} if for every \(U \in \text{cov}(Y)\) there exists a homeomorphism of \(X\) onto \(Y\) which is \(U\)-close to \(f\).

A closed subset \(A\) of \(X\) is called a \textit{Z-set} in \(X\) \((A \in Z(X))\) if \(\{f \in C(Q,X): f(Q) \cap A = \emptyset\}\) is dense in \(C(Q,X)\), where \(Q\) denotes the Hilbert cube.
(C) \( A \in Z(X) \) iff for every \( n \) the set \( \{ f \in C(I^n, X) : f(I^n) \cap A = \emptyset \} \) is dense in \( C(I^n, X) \).

We shall need the following fact [13, Lemma 4]:
(D) Let \((Y, d)\) be a metric space and \( \{ Y_n \}_{n=1}^\infty \) be a closed increasing cover of \( Y \). For \( n = 1, 2, \ldots \), let \( g_n : X \to Y \) be a surjective map from a metric space \( X \) satisfying the following conditions:

(i) \( g_n \mid g_n^{-1}(Y_n) : g_n^{-1}(Y_n) \to Y_n \) is one-to-one, and for every \( y \in Y_n \) and every neighborhood \( V \) of \( g_n^{-1}(y) \) in \( X \), there exists an open neighborhood \( U \) of \( y \) in \( Y \) with \( g_n^{-1}(U) \subset V \);
(ii) \( g_{n+1} \mid g_n^{-1}(Y_n) = g_n \mid g_n^{-1}(Y_n) \);
(iii) \( g_{n+1} \mid X \setminus g_n^{-1}(Y_n) \) is \( \alpha_n \)-close to \( g_n \mid X \setminus g_n^{-1}(Y_n) \), where \( \alpha_n(y) = 2^{-n} \{ \min 1, d(y, Y_n) \} \), with \( \alpha_0(y) = 1 \) for all \( y \).

Then the map \( g \), defined on the subset \( Z = \bigcup_{n=1}^\infty g_n^{-1}(Y_n) \) by \( g(z) = \lim g_n(z) \), is a homeomorphism of \( Z \) onto \( Y \) such that \( d(g(x), g_1(x)) < 1 \) for \( x \in Z \).

Let \( f \in C(X, Y) \), and let \( A \) be a closed subset of \( Y \). The space \((X, f)_A\) is defined to be the set \((X \setminus f^{-1}(A)) \cup A\) with the topology generated by open subsets of \( X \setminus f^{-1}(A) \) and by sets of the form \( f^{-1}(U \setminus A) \cup (U \cap A) \), where \( U \) is open in \( Y \). We define the map \( f_A : X \to (X, f)_A \) by the formula
\[
f_A(x) = \begin{cases} x & \text{for } x \in X \setminus f^{-1}(A), \\ f(x) & \text{for } x \in f^{-1}(A). \end{cases}
\]

It is an easy consequence of the definition that the function \( p_A : (X, f)_A \to Y \), defined by \( p_A f_A = f \), is continuous and satisfies the following condition:

(E) for every \( y \in A \) and every neighborhood \( V \) of \( p_A^{-1}(y) \) in \((X, f)_A\) there exists an open neighborhood \( U \) of \( y \) in \( Y \) with \( p_A^{-1}(U) \subset V \).

Let us observe that \((Y \times Z, \pi)_A\), where \( \pi : Y \times Z \to Y \) is the projection, is a cartesian product of \( Y \) and \( Z \) reduced over \( A \) (denoted by \((Y \times Z)_A\)), see [4, p. 25]. If \( X \) and \( Y \) are metrizable, then for every closed subset \( A \) of \( Y \) \((X, f)_A\) is metrizable as a subset of \((Y \times X)_A\).

3. The strong universality property for compacta. A metric ANR space is said to be strongly universal for (finite-dimensional) compacta if, for each map \( f : A \to X \) of a (finite-dimensional) compactum, each closed subset \( B \) of \( A \) such that \( f \mid B \) is an embedding, and each \( \varepsilon > 0 \), there exists an embedding \( g : A \to X \) such that \( g \) is \( \varepsilon \)-close to \( f \) and \( g \mid B = f \mid B \). The space \( l^2_2 \) is strongly universal for finite-dimensional compacta and the space \( \Sigma \) is strongly universal for compacta.

1. Lemma. Let \( X \) be a metric ANR space which is strongly universal for (finite-dimensional) compacta, let \( f : A \to X \) be a map of a compactum and let \( B \) be a closed subset of \( A \) such that \( f \mid B \) is an embedding, \( f(A \setminus B) \subset X \setminus f(B) \) (and \( A \setminus B \) is a countable union of finite-dimensional compacta). Then given \( U \in \text{cov}(X \setminus f(B)) \) there exists an embedding \( g : A \to X \) such that \( g \mid B = f \mid B \) and \( g \mid A \setminus B \) is \( U \)-close to \( f \mid A \setminus B \).

Proof. Fix a metric \( d \) on \( X \). By \( A \) there exists a continuous function \( \beta : X \setminus f(B) \to (0, 1) \)
such that every two $\beta$-close maps into $X \setminus f(B)$ are $\mathcal{U}$-close. Let $\alpha: A \to [0,1)$ be a continuous function such that $\alpha^{-1}(0) = B$ and $\alpha(a) < \beta(f(a))$ for all $a \in A \setminus B$. Let $A \setminus B = \bigcup_{n=1}^{\infty} A_n$, where $\{A_n\}_{n=1}^{\infty}$ is a increasing sequence of finite-dimensional compacta. We let $\varepsilon_n = \inf\{\alpha(a): a \in A_n\}$. Then $\{\varepsilon_n\}$ is a decreasing sequence of positive numbers with $\lim \varepsilon_n = 0$. We shall inductively construct a sequence of maps $\{f_n: A \to X\}$ such that:

(a) $f_n(A \setminus B) \subseteq X \setminus f(B)$ and $f_n|B = f|B$,
(b) $f_n|A_{n-1} = f_{n-1}|A_{n-1}$,
(c) $f_n|A_n \cup B$ is an embedding,
(d) $d(f_n(a), f_{n-1}(a)) \leq 2^{-n}\alpha(a)$ for all $a$.

We let $f_0 = f$. Assume that $f_{n-1}$ has been already constructed. Because $X \setminus f(B)$ is an ANR the restriction $h \mapsto h|A_n$ is an open map from $C(A \setminus B, X \setminus f(B))$ to $C(A_n, X \setminus f(B))$ (see [19, Lemma 1.3]). Hence, using strong universality property, we can find an embedding $v_n: A_n \to X \setminus f(B)$ such that $v_n|A_{n-1} = f_{n-1}|A_{n-1}$ and $v_n$ is so close to $f_{n-1}|A_n$ that there is an extension $g_n: A \setminus B \to X \setminus f(B)$ of $v_n$ which is $2^{-n}\varepsilon_n$-homotopic to $f_{n-1}|A \setminus B$. Let $h_n: (A \setminus B) \times [0,1] \to X \setminus f(B)$ be a $2^{-n}\varepsilon_n$-homotopy with $h_n(a,0) = f_{n-1}(a)$ and $h_n(a,1) = g_n(a)$ for $a \in A \setminus B$. Let $v_n$ be a compact neighborhood of $A_n$ in $A \setminus B$ such that

$$diam(h_n(\{a\} \times [0,1])) < 2^{-n}\alpha(a) \quad \text{for } a \in V_n.$$ 

Then the map $f_n$ defined by

$$f_n(a) = \begin{cases} h_n(a, \lambda_n(a)) & \text{for } a \in A \setminus B, \\ f(a) & \text{for } a \in B, \end{cases}$$

where $\lambda_n: A \to [0,1]$ is such that $\lambda_n^{-1}(0) \supset A \setminus V_n$ and $\lambda_n^{-1}(1) = A_n$, has the required properties. Since $A \setminus B = \bigcup_{n=1}^{\infty} A_n$, the map $g = \lim f_n$ is an embedding of $A$ onto $X$ such that $g|B = f|B$. By (d), for each $a \in A \setminus B$

$$d(f(a), g(a)) \leq \sum_{n=1}^{\infty} 2^{-n}\alpha(a) = \alpha(a) < \beta(f(a)).$$

Thus $g|A \setminus B$ is $\mathcal{U}$-close to $f|A \setminus B$.

2. LEMMA. Let $X$ be a metric ANR space. If $X$ is strongly universal for (finite-dimensional) compacta, then every (finite-dimensional) compact subset of $X$ is a $Z$-set in $X$.

PROOF. Fix a metrix $d$ on $X$. Let $X$ be strongly universal for finite-dimensional compacta and let $K$ be a finite-dimensional, compact subset of $X$. Let $f: I^n \to X$ be a map of an $n$-dimensional cube into $X$, and let $\varepsilon > 0$ be given. We shall construct a map $g: I^n \to X$ which is $\varepsilon$-close to $f$ and such that $g(I^n) \cap K = \emptyset$. By strong universality of $X$ there is an embedding $v: I^n \to X$ which is $\frac{1}{2}\varepsilon$-close to $f$. We can regard the set $B = v(I^n) \cup K$ as a subset of $I^m \times \{0\} \subset I^m \times [0,1]$, for some $m \geq n$. Then the inclusion $i: B \to X$ can be extended to a map $h: A \to X$, where $A$ is a compact neighborhood of $B$ in $I^m \times [0,1]$. By strong universality of $X$ there is an embedding $w: A \to X$ such that $w|B = i$. By compactness of $v(I^n)$ there is $t \in (0,1]$ such that $d(w(x,t), w(x,0)) < \frac{1}{2}\varepsilon$ for $x \in v(I^n)$. Let $g(y) = w(v(y), t)$ for $y \in I^n$. Then $g$ is the required map.

Analogously we can prove that every compact subset of a strongly universal for compacta, ANR space $X$ is a $Z$-set in $X$. 

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3. **Lemma.** Let $X$ be an ANR which is strongly universal for finite-dimensional compacta. Then every compact subset of $X$ which is a countable union of finite-dimensional compacta is a Z-set in $X$.

**Proof.** Let $Y$ be a complete metric space which contains $X$ and satisfies the following condition:

(3) for every compact subset $A$ in $X$, $A \in Z(X)$ iff $A \in Z(Y)$ (see [15, Proposition 4.1]).

Let $K$ be a compact subset of $X$. By Lemma 2 $K$ is a countable union of $Z$-sets in $X$. By (3) $K$ is a countable union of $Z$-sets in $Y$. Because $Y$ is complete and $K$ is closed in $Y$, $K \in Z(Y)$ (see [4, p. 151]). By (3) again $K \in Z(X)$.

4. **Near-homeomorphisms between \( \sigma \)-compacta.** A metric ANR space $X$ has the estimated extension property for compacta if, for each open subset $G$ of $X$, and each $\mathcal{U} \in \text{cov}(G)$, and each homeomorphism $v: A \to B$ between compacta in $G$ such that $v$ is $\mathcal{U}$-homotopic to $\text{id}_A$, there exists a space homeomorphism $h: X \to X$ extending $v$, and such that $h$ is $\mathcal{U}$-close to $\text{id}_X$.

4. **Theorem.** Let $X$ and $Y$ be metric ANR spaces which are countable unions of (finite-dimensional) compacta. Suppose that $X$ has the estimated extension property for compacta and $Y$ is strongly universal for (finite-dimensional) compacta. Let $f: X \to Y$ be a map with the property that, for every compactum $A$ in $Y$ and closed subset $B$ of $A$, the map $f|_A \times f^{-1}(B): X \times f^{-1}(B) \to (X,f)_A \times B$ is a near-homeomorphism. Then $f$ is a near-homeomorphism.

**Proof.** We will only consider the case when $X$ and $Y$ are countable unions of finite-dimensional compacta, and $Y$ is strongly universal for finite-dimensional compacta. Let $X = \bigsqcup_{n=1}^{\infty} A_n$ and $Y = \bigsqcup_{n=1}^{\infty} B_n$, where $A_n$ and $B_n$ are finite-dimensional compacta for $n = 1, 2, \ldots$. Let $d$ be any metric on $Y$. By (B), it is enough to check that there is a homeomorphism $h$ of $X$ onto $Y$ such that $d(h(x), f(x)) < 1$ for $x \in X$. We shall inductively construct a sequence $\{C_n\}_{n=0}^{\infty}$ of compact subsets of $Y$ and a sequence $\{h_n\}_{n=0}^{\infty}$ of homeomorphisms of $X$ onto $(X, f)_{C_n} = X_n$ such that, for $n = 1, 2, \ldots$:

(a) $C_n \supseteq B_n \cup C_{n-1}$;
(b) $h_n(A_n) \subseteq C_n$;
(c) $h_n|_{h_n^{-1}(C_n-1)} = h_{n-1}|_{h_{n-1}^{-1}(C_{n-1})}$;
(d) $p_n h_n|_{X \setminus h_n^{-1}(C_{n-1})}$ is $\alpha_n$-close to $p_{n-1} h_n^{-1} X \setminus h_{n-1}^{-1}(C_{n-1})$, where $\alpha_n: Y \setminus C_{n-1} \to (0, 1)$ is defined by $\alpha_n(y) = 2^{-n} \min\{1, d(y, C_{n-1})\}$ and $p_n: X_n \to Y$ is the map defined by $p_n f_{C_n} = f$.

We let $C_0 = \emptyset$ and $h_0 = \text{id}$. Assume that $h_i: X \to X_i$ and $X_i$ satisfying (a), (b), (c), and (d) for $0 \leq i \leq n$ have been constructed. Note that $p_n(X_n \setminus C_n) \subseteq Y \setminus C_n$. Let $\mathcal{U}$ be an open cover of $Y \setminus C_n$ such that

$V(p_n|X_n \setminus C_n, \text{st}^3 \mathcal{U}) \subseteq V(p_n|X_n \setminus C_n, \alpha_{n+1})$.

By strong universality of $Y$ and Lemma 1, there is an embedding $v$ of $D_{n+1} = h_n(A_{n+1}) \cup C_n \subset X_n$ into $Y$ such that $v|C_n = \text{id}_{C_n}$ and $v|D_{n+1} \setminus C_n$ is $\mathcal{U}$-homotopic to $p_n|D_{n+1} \setminus C_n$. Take

$C_{n+1} = B_{n+1} \cup v(D_{n+1}) \cup H((D_{n+1} \setminus C_n) \times [0, 1])$. 

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where $H: (D_{n+1} \setminus C_n) \times [0,1] \rightarrow Y$ is a $\mathcal{U}$-homotopy with $H(x,0) = p_n(x)$ and $H(x,1) = \nu(x)$ for $x \in D_{n+1} \setminus C_n$. Because $f_{C_n}|X \setminus f^{-1}(C_n)$ is a homeomorphism of $X \setminus f^{-1}(C_n)$ onto $X_n \setminus C_n$, by the assumption about the map $f$, there exists a homeomorphism $g_{n+1}: X_n \rightarrow X_{n+1}$ such that $g_{n+1}|C_n = \text{id}$ and $g_{n+1}|X_n \setminus C_n$ is $p^{-1}_{n+1}(\mathcal{U})$-homotopic to the map $f_{(C_{n+1},C_n)}|X_n \setminus C_n$, where $f_{(C_{n+1},C_n)}$ is defined by the equality $f_{(C_{n+1},C_n)} \circ f_{C_n} = f_{C_n}$. The embeddings $g_{n+1}|B_{n+1}$ and $(p_{n+1}C_{n+1})^{-1}$ are $\text{st}(p^{-1}_{n+1}(\mathcal{U}))$-homotopic. Since $X_{n+1}$, being homeomorphic to $X$, has the estimated extension property for compacta there exists a homeomorphism $u_{n+1}$ of $X_{n+1}$ onto itself which is $\text{st}^2(p^{-1}_{n+1}(\mathcal{U}))$-close to the identity and such that $u_{n+1}q_{n+1}D_{n+1} = (p_{n+1}C_{n+1})^{-1}v$. We let $h_{n+1} = u_{n+1}g_{n+1}h_n$. Then $p_{n+1}h_{n+1}$ is $\text{st}^2(\mathcal{U})$-close to $p_{n+1}g_{n+1}h_n$ and hence $p_{n+1}h_{n+1}$ is $\text{st}^3(\mathcal{U})$-close to $p_nh_n$. It is easy to see that (a)$_{n+1}$, (b)$_{n+1}$, (c)$_{n+1}$ and (d)$_{n+1}$ are satisfied.

Since each $p_n$ satisfies (E) we apply (D) to the sequences $\{C_n\}$ and $\{p_nh_n\}$. Therefore the map $h = \lim p_nh_n$ is a homeomorphism of $X$ onto $Y$ such that $d(f(x), h(x)) < 1$ for all $x \in X$.

5. Characterization of $l^\mathcal{U}_2$- and $\Sigma$-manifolds.

5. Theorem. Let $X$ be an ANR space which is a countable union of finite-dimensional compacta. Then $X$ is an $l^\mathcal{U}_2$-manifold iff it is strongly universal for finite-dimensional compacta.

Proof. By a theorem of Torunczyk [17], $X \times l^\mathcal{U}_2$ is an $l^\mathcal{U}_2$-manifold and therefore has the estimated extension property for compacta. Given an open set $U \subset X$ and a compact set $A$ in $X$ the space $(U \times l^\mathcal{U}_2)_{A \cap U}$ is an ANR (see [13, Lemma 5]). We will prove that $A \cap U$ is a $Z$-set in $(U \times l^\mathcal{U}_2)_{A \cap U}$. Take $g: I^n \rightarrow (U \times l^\mathcal{U}_2)_{A \cap U}$ and $\varepsilon > 0$. Let $\pi_U: U \times l^\mathcal{U}_2 \rightarrow (U \times l^\mathcal{U}_2)_{A \cap U}$ denote the projection. Given $\varepsilon > 0$ there exists a map $q: (U \times l^\mathcal{U}_2)_{A \times U} \times l^\mathcal{U}_2$ such that $\pi_Uq$ is $\varepsilon/2$-close to the identity (see [13, Lemma 5]). By Lemma 3 $A$ is a $Z$-set in $X$, hence $A \cap U$ is a $Z$-set in $U$ and $(A \cap U) \times l^\mathcal{U}_2$ is a $Z$-set in $U \times l^\mathcal{U}_2$ (see [4, p. 151]). Thus there exists a map $f: I^n \rightarrow U \times l^\mathcal{U}_2$ such that $f(I^n) \cap ((A \cap U) \times l^\mathcal{U}_2) = \emptyset$ and so close to $qq$ that $\pi_Uf$ is $\varepsilon/2$-close to $\pi_Uqq$. Hence $\pi_Uf$ is $\varepsilon$-close to $g$ and $\pi_Uf(I^n) \cap (A \cap U) = \emptyset$. It means that $A \cap U$ is a $Z$-set in $(U \times l^\mathcal{U}_2)_{A \cap U}$. By [15] the projection $\pi_U: (U \times l^\mathcal{U}_2)_{A \cap U} \rightarrow (U \times l^\mathcal{U}_2)_{A \cap U}$ is a near-homeomorphism. Thus the projection $\pi: X \times l^\mathcal{U}_2 \rightarrow X$ satisfies the assumption of Theorem 4. Hence $\pi$ is a near-homeomorphism and $X$ is an $l^\mathcal{U}_2$-manifold.

6. Theorem. Let $X$ be an ANR. Then $X$ is a $\Sigma$-manifold iff it is $\sigma$-compact and is strongly universal for compacta.

Proof. We can repeat the proof of Theorem 5 replacing finite-dimensional compacta by compacta and $l^\mathcal{U}_2$ by $\Sigma$.

6. Questions. Let us formulate questions which are closely related to the problem of identifying $\sigma$-compact manifolds.

Let $G$ be a locally contractible, metrizable topological group which is a countable union of finite-dimensional compacta and is not locally compact. We do not know whether $G$ must be strongly universal for finite-dimensional compacta, and therefore an $l^\mathcal{U}_2$-manifold (see [9, TCG]).
Henderson and Walsh in [11] have constructed an example of a cell-like decomposition $G$ of $l^2$ such that the decomposition space $l^2/\mathcal{G}$ is not homeomorphic to $l^2$ but $l^2/\mathcal{G} \times [0,1]$ is homeomorphic to $l^2$. Let us mention that there is no such decomposition of the Hilbert space $l_2$ (see [12]). It means that the space $l^2$ behaves more like finite-dimensional euclidean spaces than like the Hilbert space $l_2$. Hence the following question is interesting.

7. **Question.** Let $G$ be a cell-like decomposition of $l^2$ such that $l^2/\mathcal{G}$ is a countable union of finite-dimensional compacta. Is $l^2/\mathcal{G} \times [0,1]$ (or $l^2/\mathcal{G} \times [0,1]^2$) homeomorphic to $l^2$?

Because the condition (2) is difficult to verify for some spaces it would be useful to find some new conditions characterizing $l^2$-manifolds. Examples of Henderson and Walsh [11] show that the following conditions are not sufficient to assure than ANR $X$, which is a countable union of finite-dimensional compacta, is an $l^2$-manifold:

1. every compact subset of $X$ is a $Z$-set in $X$,
2. every map $f: \bigoplus_{n=1}^{\infty} I^n \to X$ of the countable, free union of finite-dimensional cubes is strongly approximable by maps $g: \bigoplus_{n=1}^{\infty} I^n \to X$, for which the collection $\{g(I^n)\}_{n=1}^{\infty}$ is discrete.

Note that there is a topologically complete separable metric AR space, which is not homeomorphic to $l_2$, but which satisfies (4) (see [2]).

Let $E$ be the class of dense linear subspaces of $l_2$ which are countable unions of compacta with defined transfinite dimension. For every $E \in E$ let $\gamma(E)$ be the infimum of ordinals $\alpha$ such that $E$ is a countable union of compacta with transfinite dimension $< \alpha$ (see [4, p. 282]).

8. **Question.** Let $E_1, E_2 \in E$ and let $\gamma(E_1) = \gamma(E_2)$. Are $E_1$ and $E_2$ homeomorphic?

9. **Question.** Is it true that for every $\alpha \in \{\gamma(E): E \in E\}$ there is $E_\alpha \in E$, with $\gamma(E) = \alpha$ and which is topologically universal for all compacta with transfinite dimension $< \alpha$?

Note that every $\sigma$-compact linear subspace $E$ of $l_2$ which is universal for compacta is homeomorphic to $\Sigma$ (see [7]).

**ADDED IN PROOF.** The proofs of Theorems 5 and 6 are based on the following theorem of Toruńczyk [15, Proposition 5.1]:

1. if $A$ is a compact $Z$-set in a $\sigma$-compact ($\sigma$-finite-dimensional compact) ANR $X$, then the projection $\pi_A: X \times E \to (X \times E)_A$ is a near homeomorphism, where $E = \Sigma$ or $l^2$.

It has been observed recently that the above theorem is false. Moreover it turns out that the strong universality properties do not characterize $l^2$- and $\Sigma$-manifolds among $\sigma$-compact ANR’s. Theorems 5 and 6 are true if, in addition, the space $X$ satisfies the following condition:

2. every compact subset $A$ of $X$ is a strong $Z$-set in $X$ (i.e. given an open cover $\mathcal{U}$ of $X$ there exists $f: X \to X$, $\mathcal{U}$-close to the identity map, such that $f(X) \cap V = \emptyset$ for some neighborhood $V$ of $A$).

Proofs are the same and use (6) which holds if $A$ is a strong $Z$-set in $X$. Let us note that for $\sigma$-compact ANR’s (7) is equivalent to (5). Details of proofs and related examples will appear in [21].
REFERENCES


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