PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 1, September 1984

REGULARITY OF THE DISTANCE FUNCTION

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ABSTRACT. A coordinate-free proof is given of the fact that the distance function δ for a C^k submanifold M of $\mathbf{R}^n$ is C^k near M when $k \geq 2$. The result holds also when $k = 1$ if M has a neighborhood with the unique nearest point property. The differentiability of δ in the C^1 case is seen to follow directly from geometric considerations.

In the study of analysis and geometry, the function that measures the distance to a submanifold plays an important role. Let M be a submanifold of $\mathbf{R}^n$, and let δ: $\mathbf{R}^n \to \mathbf{R}$ be the distance function for M, $\delta(x) = \operatorname{dist}(x, M)$. If M is C^k, then δ is easily seen to be C^{k-1} near M, since δ is always continuous and can be written in terms of the directions normal to M. It is the case, however, that δ is actually C^k near M when $k \geq 2$, and even when $k = 1$ under certain circumstances. As Krantz and Parks [**4**] point out, this fact deserves to be better known that it is.

The regularity of δ was first considered in [**1**], and the proof for the case $k \geq 2$ is found in [**2**]. The combined results (including the case $k = 1$) are given in [**4**] in a proof based on the work in [**1**].

The purpose of this note is to present a simple, coordinate-free proof of the following theorem and its C^1 analog.

THEOREM 1. *Let $M \subset \mathbf{R}^n$ be a compact, C^k submanifold with $k \geq 2$. Then M has a neighborhood U so that δ is C^k on $U - M$.*

In the C^1 case, the additional hypothesis is needed that some neighborhood of M have the *unique nearest point* property. (See [**1**, **4**].) A neighborhood U of M has this property if for every $x \in U$ there is a unique point $P(x) \in M$ so that $\delta(x) = \operatorname{dist}(x, P(x))$. The map P: $U \to M$ is called the *projection onto* M.

LEMMA. *Let M satisfy the hypothesis of Theorem 1. Then M has a neighborhood U with the unique nearest point property, and the projection map P: $U \to M$ is C^{k-1}.*

PROOF. This is just the *tubular neighborhood theorem* with the added observation that the projection P factors through the map that creates the neighborhood. (See [**3**].)

Let

$$\nu(M) = \{(p, v) \in \mathbf{R}^n \times \mathbf{R}^n \mid p \in M \text{ and } v \perp T_pM\}$$

be the normal bundle for M; it is a C^{k-1} manifold of dimension n. Define the C^{k-1} map F: $\nu(M) \to \mathbf{R}^n$ by $F(p, v) = p + v$. The Jacobian F_* is easily seen to be nonsingular along the zero section $\{(p, 0) \in \nu(M)\}$. By the inverse function theorem and the compactness of M, there is an $\varepsilon > 0$ such that F restricted to

Received by the editors September 26, 1983.
1980 *Mathematics Subject Classification*. Primary 53A07; Secondary 26B05.

$\nu_\varepsilon(M) = \{(p,v) \in \nu(M)\colon |v| < \varepsilon\}$ is a C^{k-1} diffeomorphism onto a neighborhood U of M. On U the map P is the composition

$$U \overset{F^{-1}}{\to} \nu_\varepsilon(M) \to M,$$

where the last map is projection onto the first factor. Q.E.D.

In [4] M is said to have *positive reach* of at least ε. The largest possible neighborhood U on which P is defined is determined in part by the local extrinsic geometry of M inside $\mathbf{R}^n$: the extrinsic curvature of M governs the location of the singularities of the map $F\colon \nu(M) \to \mathbf{R}^n$. (See [**5**, §6].)

PROOF OF THEOREM 1. On the neighborhood U where P is well defined, the distance function is given by $\delta(x) = \|x - P(x)\|$. For $v \in \mathbf{R}^n$, let D_v denote differentiation in the direction v. Then for $x \in U - M$,

$$(D_v\delta^2)(x) = 2(x - P(x)) \cdot (v - D_vP(x)) = 2(x - P(x)) \cdot v,$$

since $D_vP(x)$ is tangent to M. Hence

$$(\mathrm{grad}\,\delta^2)(x) = 2(x - P(x)), \tag{$*$}$$

which is C^{k-1}, and so δ is C^k on $U - M$. Q.E.D.

In the C^1 case one needs to examine the behavior of the difference quotient.

THEOREM 2. *Let M be C^1 and suppose U is a neighborhood of M with the unique nearest point property. Then δ is C^1 on $U - M$.*

PROOF. A simple argument (see [**1**, 4.8(4)]) shows that $P\colon U \to M$ is continuous. Thus, it suffices to show that $(*)$ holds on $U - M$. If this is not the case, then there is some point $x \in U - M$ and some vector $v \in \mathbf{R}^n$ such that

$$\varliminf_{t\to 0^+} \frac{\delta^2(x+tv) - \delta^2(x)}{t} < 2(x - P(x)) \cdot v \tag{1}$$

or

$$\varlimsup_{t\to 0^+} \frac{\delta^2(x+tv) - \delta^2(x)}{t} > 2(x - P(x)) \cdot v. \tag{2}$$

In the first case, one can find a fixed $\varepsilon > 0$ and then choose $t > 0$ arbitrarily close to zero such that

$$\delta^2(x+tv) < \delta^2(x) + 2(x - P(x)) \cdot tv - t\varepsilon.$$

It follows, then, that

$$\begin{aligned} \mathrm{dist}^2(x, P(x+tv)) &= \|(x + tv - P(x+tv)) - tv\|^2 \\ &= \delta^2(x+tv) - 2(x + tv - P(x+tv)) \cdot tv + t^2\|v\|^2 \\ &< \delta^2(x) + 2(P(x+tv) - P(x)) \cdot tv - t\varepsilon - t^2\|v\|^2. \end{aligned}$$

By the continuity of P, t can be chosen small enough so that

$$\mathrm{dist}(x, P(x+tv)) < \delta(x) = \mathrm{dist}(x, P(x)).$$

Then x is closer to $P(x+tv)$ than to $P(x)$, a contradiction.

Similarly, (2) leads to $\mathrm{dist}(x+tv, P(x)) < \mathrm{dist}(x+tv, P(x+tv))$. The theorem follows. Q.E.D.

REMARKS. (1) With some modifications, the same proofs will work when M is a submanifold of a Riemannian manifold.

(2) If M is a hypersurface of the form $M = \{x \in \mathbf{R}^n \mid \rho(x) = 0\}$, where ρ is a C^k function with $d\rho \neq 0$ on M, then one can form the signed distance function

$$\tilde{\delta}(x) = \begin{cases} \delta(x) & \text{for } \rho(x) \geq 0, \\ -\delta(x) & \text{for } \rho(x) \leq 0. \end{cases}$$

It is easy to see that $\tilde{\delta}$ is C^k on all of U.

(3) In the C^1 case, the regularity of M does not enter into the proof. M can be replaced by any closed set in $\mathbf{R}^n$, and U by any open set on which the projection $P: U \to M$ is well defined. (For the original treatment of this, see [**1**].)

(4) The extra hypothesis in the C^1 case is essential. The distance function for the curve $y = |x|^{3/2}$ in $\mathbf{R}^2$ is not differentiable at any point on the y-axis. See [**4**] for details.

For further remarks and examples, the reader is directed to the references, especially [**4**].

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