REMARKS ON THE PARAMETRIZED SYMBOL CALCULUS

MICHIKO KINOSHITA

ABSTRACT. In his paper, L. Hörmander has used the Weyl calculus to study the Fourier integral operator theory. In the present paper, the author considers the correspondences \( W_\tau \), \( \tau \in \mathbb{R} \) (\( \mathbb{R} \) is the set of real numbers), which mean the standard correspondence of symbol and operator if \( \tau = 0 \), and the correspondence of Weyl type if \( \tau = 1/2 \), and shows the explicit asymptotic formula which describes the deviation of \( W_\sigma (W_\tau)^{-1} \) from the automorphisms as Lie algebra, and makes some remarks on the above formula.

1. Symbol classes.

NOTATION.

\[
\rho^{(\alpha)}(x, \xi) = \partial^{\alpha}_\xi D_x^{\beta} p(x, \xi),
\]
where \( p(x, \xi) \in C^\infty(R^n_x \times R^n_\xi), \ D_x_j = -i\partial / \partial x_j, \)

\[
\rho^{(\alpha, \alpha')}_{(\beta, \beta')}(x, \xi, x', \xi') = \partial^{\alpha}_\xi \partial^{\alpha'}_{\xi'} D_x^{\beta} D_x^{\beta'} p(x, \xi, x', \xi'),
\]
where \( p(x, \xi, x', \xi') \in C^\infty(R^n_x \times R^n_\xi \times R^n_{x'} \times R^n_{\xi'}), \ \langle \xi \rangle = \sqrt{1+|\xi|^2}, \ \langle \xi; \xi' \rangle = \sqrt{1+|\xi|^2 + |\xi'|^2} \) and \( \hat{u}(\xi) \) is the Fourier transform of \( u(x) \).

We denote by \( S_n^{m, \rho, \delta} \), for any real numbers \( m, \rho, \delta \) such that \( 0 \leq \delta \leq \rho \leq 1, \delta < 1 \), the set of smooth functions \( p(x, \xi) \) on \( R^n_x \) which satisfy the condition that for any multi-indices \( \alpha, \beta \), there exists a constant \( C_{\alpha, \beta} \) such that

\[
|p^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} |\xi|^m + \delta|\beta| - \rho|\alpha|.
\]

We denote by \( S_n^{m, \rho, \delta} \), for any real numbers \( m, m', \rho, \delta \) such that \( 0 \leq \delta \leq \rho \leq 1, \delta < 1 \), the set of smooth functions \( p(x, \xi, x', \xi') \) on \( R^n_x \times R^n_\xi \times R^n_{x'} \times R^n_{\xi'} \), which satisfy the condition that for any multi-indices \( \alpha, \beta, \alpha', \beta' \), there exists a constant \( C_{\alpha, \beta, \alpha', \beta'} \) such that

\[
|p^{(\alpha, \alpha')}_{(\beta, \beta')}(x, \xi, x', \xi')| \leq C_{\alpha, \beta, \alpha', \beta'} |\xi|^m + \delta|\beta| - \rho|\alpha| + |\xi'|^m + \delta|\beta'| - \rho|\alpha'|.
\]

We denote by \( S \) the set of the rapidly decreasing functions, and we denote by \( \text{Op}(S_n^{m, \rho, \delta}) \) the set of pseudo-differential operators which is defined by

\[
(B(p)u)(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad p \in S_n^{m, \rho, \delta}, \ u \in S.
\]

In like manner, we denote by \( \text{Op}(S_n^{m, m', \rho, \delta}) \) the set of pseudo-differential operators.
which is defined by

\[(W(p)u)(x) = \int \int e^{i(x-x') \cdot \xi + ix' \cdot \xi'} p(x, \xi, x', \xi') \hat{u}(\xi') d\xi dx' d\xi', \]

\[p \in S_{\rho, \delta}^{m,m'}, \ u \in \mathcal{S}.\]

We denote by \(R^m\), for any real numbers \(m\) and \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(S_{\rho, \delta}^{m,0}\), defined by

\[(R^m_\tau(p))(x, \xi, x', \xi') = p((1-\tau)x + \tau x', \xi),\]

and we denote by \(W^m_\tau\), for any real numbers \(m\) and \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(\text{Op}(S_{\rho, \delta}^{m,0})\), defined by \(WR^m_\tau\).

REMARK 1. \(\text{Op}(S_{\rho, \delta}^{m,0}) = \text{Op}(S_{\rho, \delta}^m)\).

REMARK 2. The standard correspondence between symbols and operators is \(W^0_\tau\), and the correspondence of Weyl type is \(W^\infty_1\) (for the Weyl calculus, see [1]).

We denote by \(W^\infty_\tau\), for any real numbers \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(\text{Op}(S_{\rho, \delta}^{\infty})\), defined by \(W^\infty_\tau(p) = W^\infty_1(p)\), for any \(p \in S_{\rho, \delta}^m\).

Let \(A = W^m_\tau(p)\) and \(B = W^m_\tau(q)\), where \(p \in S_{\rho, \delta}^m\), \(q \in S_{\rho, \delta}^m\). The product \(AB\) is contained in \(\text{Op}(S_{\rho, \delta}^{m+m'})\). Therefore, there exists a function \(r \in S_{\rho, \delta}^{m+m'}\) such that \(AB = W^m_\tau(r)\). This symbol \(r\) is expressed by the formula

\[r(x, \xi, x', \xi') = \begin{pmatrix} (\xi y - x_n) (\xi y - n \xi) (H_0(\sigma) - H_0(\tau)) \\ \end{pmatrix} (x, \xi, x', \xi').\]

We use the notation \(r = p \circ_{\sigma} q\) in the following section. This notation is used in [3].

2. **Main result.** In this section we examine the linear transformation of \(\text{Op}(S_{\rho, \delta}^m)\) given by \(K^m_{\sigma, \tau} = W^m_{\sigma} (W^m_{\tau})^{-1}\), when \(\sigma \neq \tau\).

REMARK 3. The fact that \(K^\infty_{\sigma, \tau}\) is not an automorphism of \(\text{Op}(S_{\rho, \delta}^\infty)\) as algebra is reduced to the fact that \(a \circ_{\sigma} b \neq a \circ_{\tau} b\) by the composition formula above.

In this section, we consider the deviation of \(K^m_{\sigma, \tau}\) from the automorphisms of \(\text{Op}(S_{\rho, \delta}^m)\) as a Lie algebra.

REMARK 4. When \(m \leq \rho - \delta\), \(\text{Op}(S_{\rho, \delta}^m)\) is a Lie algebra. By a trivial computation, we obtain the following fact. When \(A \in \text{Op}(S_{\rho, \delta}^m)\) and \(B \in \text{Op}(S_{\rho, \delta}^m)\), we get

\[K^m_{\sigma, \tau}(A), K^m_{\sigma, \tau}(B)] - K^m_{\sigma, \tau}([A, B]) = W^m_\sigma (a \circ_{\sigma} b - b \circ_{\sigma} a - a \circ_{\tau} b + b \circ_{\tau} a),\]

where \(a = (W^m_\tau)^{-1} A\), \(b = (W^m_\tau)^{-1} B\).

We denote by \(H_n(\sigma)\) the function \(\sum_{k=1}^n F^{n-k}G^{k-1}\), where \(F = (1-\sigma)\xi \cdot y - \sigma x_n\), \(G = (1 - \sigma) x \cdot \eta - \sigma \xi \cdot y\).

REMARK 5. This function has an invariant property with respect to the changing of \(\xi \cdot y\) and \(x \cdot \eta\). Obvious calculation gives

\[H_n(\sigma) = \sum_{k=1}^n \sum_{r=0}^{n-k} (-1)^{n-k-r+s} \binom{n-k}{r} \binom{k-1}{s} \cdot (1 - \sigma)^k + r - s - 1 \sigma^{n+s-k-r} (\xi \cdot y)^{r+s} (x \cdot \eta)^{n-r-s}.\]

We denote by \(T_k(D)\) the operator corresponding to

\[T_k(x, \xi, y, \eta) = \frac{\partial^k}{\partial \xi^k} (\xi y - x_n) (H_k(\sigma) - H_k(\tau)).\]

The deviation is described by the following theorem.
THEOREM. If \( \sigma \in S_{\rho,\delta}^{m_1} \) and \( \tau \in S_{\rho,\delta}^{m_2} \), then
\[
a \circ \sigma b - b \circ \sigma a - a \circ \tau b + b \circ \tau a - \sum_{k=0}^{n} T_k(D)a(x, \xi)b(y, \eta)|(x, \xi) = (y, \eta)
\]
\[
\in S_{\rho,\delta}^{m_1 + m_2 - (n+1)(\rho - \delta)}.
\]

In the case of \( k = 0, 1, 2, 3 \), we have
\[
T_0(x, \xi, y, \eta) = 0, \quad T_1(x, \xi, y, \eta) = 0,
\]
\[
T_2(x, \xi, y, \eta) = (\sigma - \tau)(\xi y - x \eta)(\xi y + x \eta),
\]
\[
T_3(x, \xi, y, \eta) = -\frac{i}{2}(\sigma - \tau)(\xi y - x \eta)(\sigma + \tau - 1)(\xi y + x \eta)^2.
\]

PROOF. Essentially the product formula and calculations give the proof.

REMARK 6. \( T_k(x, \xi, y, \eta) \) is divisible by \( \sigma - \tau \)(\xi y - x \eta). Consequently
\[
T_k(x, \xi, y, \eta) = (\sigma - \tau)(\xi y - x \eta)U_k(x, \xi, y, \eta, \sigma, \tau),
\]
where \( U_k \) is a symmetric function of \( \xi \cdot y \) and \( x \cdot \eta \), and also a symmetric function of \( \sigma \) and \( \tau \). By the fact that \( T_2 \neq 0 \), we obtain that \( K_{\sigma,\tau}^m, m \leq \rho - \delta \), is not an automorphism of the Lie algebra \( \text{Op}(S_{\rho,\delta}^m) \). By the fact that \( T_0 = T_1 = 0 \), we obtain that \( K_{\sigma,\tau}^m, m \leq \rho - \delta \), which is induced by \( K_{\sigma,\tau}^m, m \leq \rho - \delta \), is an automorphism of the Lie algebra \( \text{Op}(S_{\rho,\delta}^m)/\text{Op}(S_{\rho,\delta}^{2m - 2(\rho - \delta)}) \).

REMARK 7. From the property that \( H_k(1 - \sigma) = (-1)^{k-1}H_k(\sigma) \), we obtain that if \( \sigma + \tau = 1 \) and \( k \) is odd, then \( T_k(x, \xi, y, \eta) = 0 \).

REFERENCES